

THE FRACTIONAL NONLOCAL ORNSTEIN–UHLENBECK EQUATION, GAUSSIAN SYMMETRIZATION AND REGULARITY

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ABSTRACT. For $0 < s < 1$, we consider the Dirichlet problem for the fractional nonlocal Ornstein–Uhlenbeck equation

$$\begin{cases} (-\Delta + x \cdot \nabla)^s u = f, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases}$$

where Ω is a possibly unbounded open subset of \mathbb{R}^n , $n \geq 2$. The appropriate functional settings for this nonlocal equation and its corresponding extension problem are developed. We apply Gaussian symmetrization techniques to derive a concentration comparison estimate for solutions. As consequences, novel L^p and $L^p(\log L)^\alpha$ regularity estimates in terms of the source term f are obtained by comparing u with half-space solutions.

1. INTRODUCTION

In the present paper we are interested in developing Gaussian symmetrization techniques and, as consequences, to obtain novel L^p and $L^p(\log L)^\alpha$ regularity estimates for solutions to nonlocal equations driven by fractional powers of the Ornstein–Uhlenbeck (OU for short) operator subject to homogeneous Dirichlet boundary conditions. More precisely, we focus on problems of the form

$$(1.1) \quad \begin{cases} (-\Delta + x \cdot \nabla)^s u = f, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \quad \text{for } 0 < s < 1,$$

where Ω is an open subset of \mathbb{R}^n with $\gamma(\Omega) < 1$. Here γ the Gaussian measure on \mathbb{R}^n , see (1.3).

Our problem (1.1) corresponds to a Markov process. Indeed, there is a stochastic process Y_t having as generator the fractional OU operator (1.1) with homogeneous Dirichlet boundary condition. The process can be obtained as follows. We first kill an OU process X_t at τ_Ω , the first exit time of X_t from the domain Ω . Let us denote the killed OU process by X_t^Ω . Then we subordinate the killed OU process X_t^Ω with an s -stable subordinator T_t . Thus $Y_t = X_{T_t}^\Omega$ is the resulting process (see for instance [6]). As explained in [16], (1.1) also arises in the context of nonlinear elasticity as the Signorini problem or the thin obstacle problem. Nonlocal equations with fractional powers of the OU operator in $\Omega = \mathbb{R}^n$ have been studied in the past. Indeed, a Harnack inequality for nonnegative solutions was proved in [42]. Fractional isoperimetric problems and semilinear equations in infinite dimensions (Wiener space) have been considered in [34] and [35]. Fractional functional inequalities were recently analyzed in [15].

The symmetrization techniques in elliptic and parabolic PDEs are nowadays very classical and efficient tools to derive optimal *a priori* estimates for solutions. The investigation in such direction started with the fundamental paper by H. Weinberger [48], see also [31]. The ideas were later fully formalized by G. Talenti in [43] for the homogeneous Dirichlet problem associated to a linear equation in divergence form with zero order term on a bounded domain of \mathbb{R}^n . In particular, [43] establishes a strong pointwise comparison between the Schwarz spherical rearrangement of the solution $u(x)$ to the original problem, and the unique radial solution $v(|x|)$ of a suitable elliptic problem defined on a ball having the same measure as the original domain and radial data. In turn, this kind of result allows to obtain regularity estimates of solutions with optimal constants. When dealing with parabolic

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equations, any form of pointwise comparison between the solution $u(x, t)$ of an initial boundary value problem and the solution $v(|x|, t)$ of a related radial problem with respect to x is in general no longer available. Indeed, in this case a weaker comparison result in the integral form, the so-called mass concentration comparison (or comparison of concentrations), holds for all times $t > 0$, see for instance [7, 32]. For a detailed survey on this theory we refer the interested reader to [44].

Quite recently, symmetrization techniques have been successfully applied to a class of fractional nonlocal equations. More precisely, results in terms of symmetrization were obtained for equations driven by the fractional Dirichlet Laplacian

$$(-\Delta_D)^s u = f,$$

and by the fractional Neumann Laplacian

$$(-\Delta_N)^s u = f,$$

in bounded domains of \mathbb{R}^n , for $0 < s < 1$. These equations arise in several important applications, see for example [2, 16, 39, 41]. The fractional operators above are defined in terms of the corresponding eigenfunction expansions. Then the characterization provided by the extension problem of [40] via the Dirichlet-to-Neumann map for a (degenerate or singular) elliptic PDE allows to treat the above-mentioned problems with local techniques (we also refer the reader to [14] for the fractional Laplacian on \mathbb{R}^n and to [26] for the most general extension result available, namely, for infinitesimal generators of integrated semigroups in Banach spaces). This information was essential to start a program regarding the applications of symmetrization in PDEs with fractional Laplacians. Indeed, the first paper in such direction was the seminal work [21] for the case of the fractional Dirichlet Laplacian. Those ideas were extended and enriched with many other applications to nonlinear fractional parabolic equations in [38, 45, 46]. When Neumann boundary conditions in fractional elliptic and parabolic problems are assumed, the symmetrization tools applied to the extension problem still lead to a comparison result, though of a different type, see [47].

It is important to notice that all the comparison results in the nonlocal setting we just mentioned are not pointwise in nature, but in the form of mass concentration comparison. One motivation of such phenomenon relies on the fact that the symmetrization argument applies on the extension problem with respect to the spatial variable x , by freezing the extra extension variable $y > 0$. In other words, a comparison of the solution to the extension problem is given in terms of the so-called Steiner symmetrization.

On the other hand, for elliptic equations involving the OU operator

$$\mathcal{L} = -\Delta + x \cdot \nabla,$$

the first comparison result through symmetrization, in the pointwise form, was obtained in [9]. The symmetrization has to take into account the natural variational structure of the OU operator. Indeed, the Dirichlet problem for \mathcal{L} is of the form

$$(1.2) \quad \begin{cases} -\operatorname{div}(\varphi \nabla u) = f\varphi, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases}$$

where $\varphi = \varphi(x)$ is the density of the Gaussian measure $d\gamma$ with respect to the Lebesgue measure:

$$(1.3) \quad d\gamma(x) = \varphi(x) dx = (2\pi)^{-n/2} \exp(-|x|^2/2) dx, \quad \text{for } x \in \mathbb{R}^n.$$

The source term f is then taken in the suitable class of weighted L^p spaces. Moreover, the meaningful case is when Ω is an unbounded open set. Here we assume

$$\gamma(\Omega) < 1.$$

Hence, the comparison result must be done through *Gaussian symmetrization* instead of the usual Schwarz symmetrization. In this setting, one of the main tools in the proof is the Gaussian isoperimetric inequality, which states that among all measurable subsets of \mathbb{R}^n with prescribed Gaussian measure, the half-space is the minimizer of the Gaussian perimeter. It becomes rather intuitive to guess that the Schwarz spherical rearrangement of a function (which is a special radial, decreasing function), appearing in the comparison results in the Lebesgue setting, should now be replaced by the

rearrangement with respect to the Gaussian measure. The latter is a particular increasing function, depending only on one variable, defined in a half-space (see Subsection 2.3 for definitions and related properties). The authors of [9] were able to apply this powerful machinery to compare the solution u (in the sense of rearrangement) to (1.2) with the solution v to the problem

$$(1.4) \quad \begin{cases} -\operatorname{div}(\varphi \nabla v) = f^\star \varphi, & \text{in } \Omega^\star \\ v = 0, & \text{on } \partial\Omega^\star, \end{cases}$$

where Ω^\star is a half-space having the same Gaussian measure as Ω and f^\star is the n -dimensional Gaussian rearrangement of f . The solution v to (1.4) (parallel to the classical case described in [43]) can be explicitly written, allowing the authors to derive the sharp a priori pointwise estimate

$$u^\star(x) \leq v(x), \quad \text{for } x \in \Omega^\star.$$

This was the starting point to obtain regularity results for u in Lorentz-Zygmund spaces. Generalizations of this result for elliptic and parabolic problems involving elliptic operators in divergence form which are degenerate with respect to the Gaussian measure are contained in [17, 20], see also references therein.

Our main concern is to get sharp estimates for the solution u to (1.1) by comparing it with the solution ψ to the problem

$$(1.5) \quad \begin{cases} \mathcal{L}^s \psi = f^\star, & \text{in } \Omega^\star, \\ \psi = 0, & \text{on } \partial\Omega^\star. \end{cases}$$

As our previous discussion evidences, (1.5) is actually a one dimensional problem. Our idea that yields the desired result reads as follows. Using the main extension result of [40] we can characterize the fractional OU operator \mathcal{L}^s in (1.1) as a suitable Dirichlet-to-Neumann map. This allows us to obtain the solution u to (1.1) as the trace on Ω of the solution $w = w(x, y)$ of the following degenerate elliptic boundary value problem, which will be called the *extension problem* associated to (1.1):

$$(1.6) \quad \begin{cases} -\operatorname{div}(y^a \varphi(x) \nabla_{x,y} w) = 0, & \text{in } \mathcal{C}_\Omega, \\ w = 0, & \text{on } \partial_L \mathcal{C}_\Omega, \\ -\lim_{y \rightarrow 0^+} y^a w_y = f, & \text{on } \Omega. \end{cases}$$

Here

$$(1.7) \quad a := 1 - 2s \in (-1, 1),$$

while

$$\mathcal{C}_\Omega := \Omega \times (0, \infty)$$

is the infinite cylinder of basis Ω , and $\partial_L \mathcal{C}_\Omega := \partial\Omega \times [0, \infty)$ is its lateral boundary. In a similar way, the solution ψ to (1.5) can be seen as the trace over Ω^\star of the solution $v = v(x, y)$ to

$$(1.8) \quad \begin{cases} -\operatorname{div}(y^a \varphi(x) \nabla_{x,y} v) = 0, & \text{in } \mathcal{C}_\Omega^\star, \\ v = 0, & \text{on } \partial_L \mathcal{C}_\Omega^\star, \\ -\lim_{y \rightarrow 0^+} y^a v_y = f^\star, & \text{on } \Omega^\star, \end{cases}$$

where

$$(1.9) \quad \mathcal{C}_\Omega^\star := \Omega^\star \times (0, \infty),$$

and $\partial_L \mathcal{C}_\Omega^\star := \partial\Omega^\star \times [0, \infty)$. Therefore, the problem reduces to look for a *mass concentration comparison* between the solution w to (1.6) and the solution v to (1.8). More precisely, we prove that

$$(1.10) \quad \int_0^r w^\otimes(\sigma, y) d\sigma \leq \int_0^r v^\otimes(\sigma, y) d\sigma, \quad \text{for all } r \in [0, \gamma(\Omega)],$$

where, for all $y \geq 0$, the functions $w^\otimes(\cdot, y)$ and $v^\otimes(\cdot, y)$ are the one dimensional Gaussian rearrangements of w and v , respectively. The key role of this framework is played by a novel second order

derivation formula for functions defined by integrals, see Corollary 2.13, whose proof presents new nontrivial technical difficulties owed to the Gaussian framework. As a consequence, we will obtain L^p and $L^p(\log L)^\alpha$ estimates for u in terms of f .

The paper is organized as follows. Section 2 contains the preliminaries needed for the developments of our results. In particular, we briefly describe some basic properties of the Gaussian measure and the OU semigroup. Moreover, we carefully develop a full and self-contained analysis of the main functional setting where problems (1.1) and (1.6) are posed. Section 2 ends with the introduction of the basic definitions and properties of symmetrization with respect to the Gaussian measure. In this regard, we will present the proof of the derivation formula stated in Theorem 2.12, whose consequence is the above-mentioned second order differentiation formula, see Corollary 2.13. Section 3 is entirely devoted to the proof of the comparison (1.10), that is, our main result Theorem 3.1. In Section 4 we present our novel Gaussian–Zygmund $L^p(\log L)^\alpha(\Omega, \gamma)$ and $L^p(\Omega, \gamma)$ regularity estimates for solutions u in terms of the datum f , see Theorem 4.3. More precisely, our main result (Theorem 3.1) is combined with $L^p(\log L)^\alpha$ regularity estimates of the solution ψ to problem (1.5), which is obtained by using the explicit form of ψ in terms of the fractional integral $\mathcal{L}^{-s}(f^*)$ and the OU semigroup. Finally, in the Appendix we shall use suitable estimates of the Mehler kernel to exhibit a semigroup language proof of the regularity estimates when the datum f belongs to the smaller Gaussian–Lebesgue space $L^p(\Omega, \gamma)$.

2. PRELIMINARIES, FUNCTIONAL SETTING, AND THE SECOND ORDER DERIVATION FORMULA

In this section we recall the basic tools we are going to use in the proof of our main comparison result, Theorem 3.1, and its consequences. First, we introduce some basics about Gaussian analysis and the OU semigroup. Then the necessary functional background to precise the fractional nonlocal equations (1.1) and (1.5), and their extension problems (1.6) and (1.8) will be developed. Finally, after presenting definitions and properties of rearrangement techniques in the Gaussian framework, we will prove our novel second order derivation formula, see Theorem 2.12 and Corollary 2.13.

2.1. Gaussian analysis and the OU semigroup.

2.1.1. Gaussian measure and isoperimetry. Let $d\gamma$ be the n -dimensional normalized Gaussian measure on \mathbb{R}^n defined in (1.3). Let Ω be an open subset of \mathbb{R}^n , possibly unbounded. We denote by $H^1(\Omega, \gamma)$ the Sobolev space with respect to the Gaussian measure, which is obtained as the completion of $C^\infty(\overline{\Omega})$ with respect to the norm

$$\|u\|_{H^1(\Omega, \gamma)}^2 = \int_{\Omega} u^2 d\gamma(x) + \int_{\Omega} |\nabla u|^2 d\gamma(x).$$

By $H_0^1(\Omega, \gamma)$ we denote the closure of $C_c^\infty(\Omega)$ in the norm of $H^1(\Omega, \gamma)$. The following Poincaré inequality holds (see for instance [22]): if $\gamma(\Omega) < 1$ then there exists a constant $C_\Omega > 0$ such that

$$(2.1) \quad \int_{\Omega} |u|^2 d\gamma(x) \leq C_\Omega \int_{\Omega} |\nabla u|^2 d\gamma(x), \quad \text{for all } u \in H_0^1(\Omega, \gamma).$$

One of the main tools to prove the comparison result is the *Gaussian isoperimetric inequality*. Let us define the perimeter with respect to Gaussian measure as

$$P(E) = \int_{\partial E} \varphi(x) d\mathcal{H}^{n-1}(x),$$

where E is an $(n-1)$ -rectifiable set and \mathcal{H}^{n-1} denotes the $(n-1)$ -dimensional Hausdorff measure. It is well known (see [13]) that among all measurable sets of \mathbb{R}^n with prescribed Gaussian measure, the half-spaces take the smallest perimeter. More precisely, we have

$$(2.2) \quad P(E) \geq \frac{1}{\sqrt{2\pi}} \exp(-[\Phi^{-1}(\gamma(E))]^2/2),$$

for all subsets $E \subset \mathbb{R}^n$, where, for $\lambda \in \mathbb{R} \cup \{-\infty, +\infty\}$, we set

$$(2.3) \quad \Phi(\lambda) := \frac{1}{\sqrt{2\pi}} \int_{\lambda}^{\infty} e^{-r^2/2} dr.$$

2.1.2. *The OU semigroup.* We recall some remarkable properties of the OU semigroup (see [5, 11] for further details) which will turn out to be useful in the following.

The solution to the Cauchy problem

$$\begin{cases} \rho_t + \mathcal{L}\rho = 0, & \text{in } \mathbb{R}^n \times (0, \infty), \\ \rho(x, 0) = g(x), & \text{on } \mathbb{R}^n, \end{cases}$$

is given by the OU semigroup

$$\rho(x, t) = e^{-t\mathcal{L}}g(x).$$

It is a classical fact that such a semigroup can be expressed in terms of a suitable integral kernel. More precisely, if $g \in L^p(\mathbb{R}^n, \gamma)$, for $1 \leq p \leq \infty$, then

$$(2.4) \quad e^{-t\mathcal{L}}g(x) = \int_{\mathbb{R}^n} M_t(x, y)g(y) d\gamma(y), \quad \text{for } x \in \mathbb{R}^n, t > 0.$$

Here $M_t(x, y)$ is the so-called Mehler kernel, which is defined by

$$(2.5) \quad M_t(x, y) = \frac{1}{(1 - e^{-2t})^{n/2}} \exp\left(-\frac{e^{-2t}|x|^2 - 2e^{-t}\langle x, y \rangle + e^{-2t}|y|^2}{2(1 - e^{-2t})}\right).$$

We recall that

$$(2.6) \quad \int_{\mathbb{R}^n} M_t(x, y) d\gamma(y) = 1, \quad \text{for all } x \in \mathbb{R}^n, t > 0,$$

and that if $g \in L^p(\mathbb{R}^n, \gamma)$, $1 \leq p < \infty$, then

$$(2.7) \quad \|e^{-t\mathcal{L}}g\|_{L^p(\mathbb{R}^n, \gamma)} = \left\| \int_{\mathbb{R}^n} M_t(\cdot, y)g(y) d\gamma(y) \right\|_{L^p(\mathbb{R}^n, \gamma)} \leq \|g\|_{L^p(\mathbb{R}^n, \gamma)}.$$

It is standard to define the OU semigroup on a domain Ω of \mathbb{R}^n subject to homogenous Dirichlet boundary conditions. Indeed, the solution to the Cauchy-Dirichlet problem

$$(2.8) \quad \begin{cases} \eta_t + \mathcal{L}\eta = 0, & \text{in } \Omega \times (0, \infty), \\ \eta(x, t) = 0, & \text{on } \partial\Omega \times [0, \infty), \\ \eta(x, 0) = f(x), & \text{on } \Omega, \end{cases}$$

is given by the semigroup generated by the OU in Ω with Dirichlet boundary conditions:

$$\eta(x, t) = e^{-t\mathcal{L}_\Omega}f(x).$$

It follows from standard parabolic regularity theory that η is smooth in $\Omega \times (0, \infty)$. Now, let us choose $\Omega = H$, where H is the half-space $H := \{x = (x_1, x') \in \mathbb{R}^n : x_1 > 0, x' \in \mathbb{R}^{n-1}\}$ and define

$$(2.9) \quad \tilde{f}(x) = \begin{cases} f(x_1, x'), & \text{for } x \in H, \\ -f(-x_1, x'), & \text{for } x \in \mathbb{R}^n \setminus H. \end{cases}$$

Observe that for $1 \leq p < \infty$ we have

$$(2.10) \quad \|\tilde{f}\|_{L^p(\mathbb{R}^n, \gamma)} = 2\|f\|_{L^p(H, \gamma)}.$$

It is not difficult to check (see for example [36]) that in this case the semigroup associated to (2.8) is obtained as the restriction to H of the OU semigroup on \mathbb{R}^n applied to \tilde{f} , that is,

$$(2.11) \quad \eta(x, t) = e^{-t\mathcal{L}_H}f(x) = e^{-t\mathcal{L}}\tilde{f}(x)|_H.$$

Moreover, using the expression of the OU semigroup in terms of the Mehler kernel (2.4) we see that the following explicit formula holds in dimension $n = 1$:

$$(2.12) \quad \eta(x, t) = \int_0^\infty [M_t(x, y) - M_t(x, -y)] f(y) d\gamma(y), \quad \text{for all } x > 0, t > 0.$$

2.2. The fractional nonlocal OU equation and the extension problem. We introduce now an appropriate functional setting, which is essential when dealing with problems (1.1) and (1.6). In order to define the fractional powers $\mathcal{L}^s u$, $0 < s < 1$, we consider the sequence of eigenvalues $0 < \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_k \nearrow \infty$ and the corresponding orthonormal basis of Dirichlet eigenfunctions $\{\psi_k\}_{k \geq 1}$ of \mathcal{L} in $L^2(\Omega, \gamma)$, see for example [10]. In other words, for every $k \geq 1$, $\psi_k \in L^2(\Omega, \gamma)$ is a weak solution to the Dirichlet problem

$$\begin{cases} -\operatorname{div}(\varphi \nabla \psi_k) = \lambda_k \varphi \psi_k, & \text{in } \Omega, \\ \psi_k = 0, & \text{on } \partial\Omega. \end{cases}$$

Now, let us define the Hilbert space

$$\mathcal{H}^s(\Omega, \gamma) \equiv \operatorname{Dom}(\mathcal{L}^s) := \left\{ u \in L^2(\Omega, \gamma) : \sum_{k=1}^{\infty} \lambda_k^s |\langle u, \psi_k \rangle_{L^2(\Omega, \gamma)}|^2 < \infty \right\},$$

with scalar product

$$\langle u, v \rangle_{\mathcal{H}^s(\Omega, \gamma)} := \sum_{k=1}^{\infty} \lambda_k^s \langle u, \psi_k \rangle_{L^2(\Omega, \gamma)} \langle v, \psi_k \rangle_{L^2(\Omega, \gamma)}.$$

Then the norm in $\mathcal{H}^s(\Omega, \gamma)$ is given by

$$\|u\|_{\mathcal{H}^s(\Omega, \gamma)}^2 = \sum_{k=1}^{\infty} \lambda_k^s |\langle u, \psi_k \rangle_{L^2(\Omega, \gamma)}|^2.$$

For $u \in \mathcal{H}^s(\Omega, \gamma)$, we define $\mathcal{L}^s u$ as the element in the dual space $(\mathcal{H}^s(\Omega, \gamma))'$ through the formula

$$\mathcal{L}^s u = \sum_{k=1}^{\infty} \lambda_k^s \langle u, \psi_k \rangle_{L^2(\Omega, \gamma)} \psi_k, \quad \text{in } (\mathcal{H}^s(\Omega, \gamma))'.$$

That is, for any function $v \in \mathcal{H}^s(\Omega, \gamma)$ we have

$$\langle \mathcal{L}^s u, v \rangle = \sum_{k=1}^{\infty} \lambda_k^s \langle u, \psi_k \rangle_{L^2(\Omega, \gamma)} \langle v, \psi_k \rangle_{L^2(\Omega, \gamma)} = \langle u, v \rangle_{\mathcal{H}^s(\Omega, \gamma)}.$$

This identity can be rewritten as

$$\langle \mathcal{L}^s u, v \rangle = \int_{\Omega} (\mathcal{L}^{s/2} u)(\mathcal{L}^{s/2} v) dx, \quad \text{for every } u, v \in \mathcal{H}^s(\Omega, \gamma),$$

where $\mathcal{L}^{s/2}$ is defined by taking the power $s/2$ of the eigenvalues λ_k .

Remark 2.1 (The fractional OU operator is a nonlocal operator). *By using the language of semigroups as in [40], see also [16, 41, 42], it can be seen that the fractional operator \mathcal{L}^s is a nonlocal operator. Indeed, we have the semigroup and kernel formulas*

$$\begin{aligned} \mathcal{L}^s u(x) &= \frac{1}{\Gamma(-s)} \int_0^\infty (e^{-t\mathcal{L}^\Omega} u(x) - u(x)) \frac{dt}{t^{1+s}} \\ &= \operatorname{PV} \int_{\Omega} (u(x) - u(y)) K_s(x, y) dy + u(x) B_s(x), \end{aligned}$$

where PV means that the integral is taken in the principal value sense. Here

$$e^{-t\mathcal{L}^\Omega} u(x) = \int_{\Omega} H_t(x, y) u(y) d\gamma(y),$$

is the semigroup generated by \mathcal{L} in Ω with Dirichlet boundary conditions, $H_t(x, y)$ is the corresponding heat kernel,

$$K_s(x, y) = \frac{1}{|\Gamma(-s)|} \int_0^\infty H_t(x, y) \frac{dt}{t^{1+s}}, \quad x, y \in \Omega,$$

and

$$B_s(x) = \frac{1}{|\Gamma(-s)|} \int_0^\infty (1 - e^{-t\mathcal{L}^\Omega} 1(x)) \frac{dt}{t^{1+s}}, \quad x \in \Omega.$$

In the particular case of $\Omega = \mathbb{R}^n$, we have $H_t(x, y) = M_t(x, y)$, the Mehler kernel, and, as a direct consequence of (2.6), we see that $B_s(x) \equiv 0$. Though this description is important, we will not use it here. Instead, we will apply the extension technique.

Recalling the notation in (1.7), we define the Sobolev energy space on the infinite cylinder \mathcal{C}_Ω :

$$H_{0,L}^1(\mathcal{C}_\Omega, d\gamma(x) \otimes y^a dy) = \left\{ v \in H_{\text{loc}}^1(\mathcal{C}_\Omega) : v = 0 \text{ on } \partial_L \mathcal{C}_\Omega, \int \int_{\mathcal{C}_\Omega} y^a (v^2 + |\nabla_{x,y} v|^2) d\gamma(x) dy < \infty \right\}.$$

By the Gaussian Poincaré inequality (2.1), for each $v \in H_{0,L}^1(\mathcal{C}_\Omega, d\gamma(x) \otimes y^a dy)$ we have

$$\begin{aligned} \int \int_{\mathcal{C}_\Omega} y^a v^2 d\gamma(x) dy &= \int_0^\infty y^a \int_\Omega v^2 d\gamma(x) dy \leq C_\Omega \int_0^\infty y^a \int_\Omega |\nabla_x v|^2 d\gamma(x) dy \\ &\leq C_\Omega \int \int_{\mathcal{C}_\Omega} y^a |\nabla_{x,y} v|^2 d\gamma(x) dy. \end{aligned}$$

Thus we can equip the space $H_{0,L}^1(\mathcal{C}_\Omega, d\gamma(x) \otimes y^a dy)$ with the equivalent norm

$$\|v\|_{H_{0,L}^1(\mathcal{C}_\Omega, d\gamma(x) \otimes y^a dy)}^2 = \int \int_{\mathcal{C}_\Omega} y^a |\nabla_{x,y} v|^2 d\gamma(x) dy,$$

which is actually the norm defined through the scalar product

$$\langle v, w \rangle_{H_{0,L}^1(\mathcal{C}_\Omega, d\gamma(x) \otimes y^a dy)} = \int \int_{\mathcal{C}_\Omega} y^a \nabla_{x,y} v \cdot \nabla_{x,y} w d\gamma(x) dy.$$

Furthermore, since we can identify $H_{0,L}^1(\mathcal{C}_\Omega, d\gamma(x) \otimes y^a dy)$ with the space $H^1((0, \infty), y^a dy; H_0^1(\Omega, \gamma))$, we have that $H_{0,L}^1(\mathcal{C}_\Omega, d\gamma(x) \otimes y^a dy)$ is a Hilbert space.

The following Theorem is a particular case of [40, Theorem 1.1], see also [16, 26, 42]. It provides the characterization of $\mathcal{L}^s u$ as the Dirichlet-to-Neumann map for a degenerate elliptic extension problem in the upper cylinder \mathcal{C}_Ω , for any $u \in \mathcal{H}^s(\Omega, \gamma)$. As the solution $w(x, y)$ is explicitly given by (2.13) and (2.16), the proof is just a verification of the statements, see for example [40, 41].

Theorem 2.2 (Extension problem). *Let $u \in \mathcal{H}^s(\Omega, \gamma)$. Define*

$$(2.13) \quad w(x, y) \equiv \mathcal{P}_y^s u(x) = \frac{2^{1-s}}{\Gamma(s)} \sum_{k=1}^{\infty} (\lambda_k^{1/2} y)^s \mathcal{K}_s(\lambda_k^{1/2} y) \langle u, \psi_k \rangle_{L^2(\Omega, \gamma)} \psi_k(x),$$

for $y \geq 0$, where \mathcal{K}_s is the modified Bessel function of the second kind and order $0 < s < 1$. Then $w \in H_{0,L}^1(\mathcal{C}_\Omega, d\gamma(x) \otimes y^a dy)$ and it is the unique weak solution to the extension problem

$$(2.14) \quad \begin{cases} -\operatorname{div}_{x,y}(y^a \varphi(x) \nabla_{x,y} w) = 0, & \text{in } \mathcal{C}_\Omega, \\ w = 0, & \text{on } \partial_L \mathcal{C}_\Omega, \\ w(x, 0) = u(x), & \text{on } \Omega, \end{cases}$$

that vanishes weakly as $y \rightarrow \infty$. More precisely,

$$\int \int_{\mathcal{C}_\Omega} y^a (\nabla_{x,y} w \cdot \nabla_{x,y} \xi) d\gamma(x) dy = 0,$$

for all test functions $\xi \in H_{0,L}^1(\mathcal{C}_\Omega, d\gamma(x) \otimes y^a dy)$ with zero trace over Ω , $\operatorname{tr}_\Omega \xi = 0$, and

$$\lim_{y \rightarrow 0^+} w(x, y) = u(x)$$

in $L^2(\Omega, \gamma)$. Furthermore, the function w is the unique minimizer of the energy functional

$$(2.15) \quad \mathcal{F}(v) = \frac{1}{2} \int \int_{\mathcal{C}_\Omega} y^a |\nabla_{x,y} v|^2 d\gamma(x) dy,$$

over the set $\mathcal{U} = \{v \in H_{0,L}^1(\mathcal{C}_\Omega, d\gamma(x) \otimes y^a dy) : \text{tr}_\Omega v = u\}$. We can also write

$$(2.16) \quad w(x, y) = \frac{y^{2s}}{4^s \Gamma(s)} \int_0^\infty e^{-y^2/(4t)} e^{-t\mathcal{L}_\Omega} u(x) \frac{dt}{t^{1+s}}.$$

Moreover,

$$-\lim_{y \rightarrow 0^+} y^a w_y = c_s \mathcal{L}^s u, \quad \text{in } (\mathcal{H}^s(\Omega, \gamma))',$$

where $c_s = \frac{\Gamma(1-s)}{4^{s-1/2}\Gamma(s)} > 0$. Finally, the following energy identity holds:

$$(2.17) \quad \iint_{\mathcal{C}_\Omega} y^a |\nabla_{x,y} w|^2 d\gamma(x) dy = c_s \|\mathcal{L}^{s/2} u\|_{L^2(\Omega, \gamma)}^2.$$

Theorem 2.2 shows in particular that the domain $\mathcal{H}^s(\Omega, \gamma)$ is contained in the range of the trace operator on $H_{0,L}^1(\mathcal{C}_\Omega, d\gamma(x) \otimes y^a dy)$ at $y = 0$. The next Lemma shows that actually these two spaces coincide.

Lemma 2.3 (Trace inequality). *We have*

$$\text{tr}_\Omega(H_{0,L}^1(\mathcal{C}_\Omega, d\gamma(x) \otimes y^a dy)) = \mathcal{H}^s(\Omega, \gamma).$$

Moreover, for all $v \in H_{0,L}^1(\mathcal{C}_\Omega, d\gamma(x) \otimes y^a dy)$,

$$(2.18) \quad \|\mathcal{L}^{s/2} v(x, 0)\|_{L^2(\Omega, \varphi)}^2 \leq (2c_s)^{-1} \iint_{\mathcal{C}_\Omega} y^a |\nabla_{x,y} v|^2 d\gamma(x) dy.$$

In particular, equality holds in (2.18) if $v = \mathcal{P}_y^s(\text{tr}_\Omega v)(x)$, (see (2.13)).

Proof. Let $u = \text{tr}_\Omega v$, for $v \in H_{0,L}^1(\mathcal{C}_\Omega, d\gamma(x) \otimes y^a dy)$ and define the function w as in (2.13). It is readily checked that w satisfies (2.14), so it minimizes the functional \mathcal{F} in (2.15). Therefore, by (2.17), $\|\mathcal{L}^{s/2} u\|_{L^2(\Omega, \gamma)}^2 \leq (c_s)^{-1} \|v\|_{H_{0,L}^1(\mathcal{C}_\Omega, d\gamma(x) \otimes y^a dy)}^2 < \infty$, that is, $u \in \mathcal{H}^s(\Omega, \gamma)$. Now (2.18) is clear. \square

Proposition 2.4 (Compactness of the trace embedding). *We have*

$$\text{tr}_\Omega(H_{0,L}^1(\mathcal{C}_\Omega, d\gamma(x) \otimes y^a dy)) \subset \subset L^2(\Omega, \gamma).$$

Proof. We need to check that the trace operator $\text{tr}_\Omega : H_{0,L}^1(\mathcal{C}_\Omega, d\gamma(x) \otimes y^a dy) \rightarrow L^2(\Omega, \gamma)$ is compact. It is clear that tr_Ω is continuous from $H_{0,L}^1(\mathcal{C}_\Omega, d\gamma(x) \otimes y^a dy)$ into $L^2(\Omega, \gamma)$ since (2.18) holds. Similarly, the finite rank operators T_j , $j \geq 1$, defined by

$$T_j v = \sum_{k=1}^j \langle v(\cdot, 0), \psi_k \rangle_{L^2(\Omega, \gamma)} \psi_k,$$

are continuous from $H_{0,L}^1(\mathcal{C}_\Omega, d\gamma(x) \otimes y^a dy)$ into $L^2(\Omega, \gamma)$. By using (2.18) and the fact that $\lambda_k \nearrow \infty$, as $k \rightarrow \infty$, we see that, if $v \in H_{0,L}^1(\mathcal{C}_\Omega, d\gamma(x) \otimes y^a dy)$,

$$\begin{aligned} \|T_j v - \text{tr}_\Omega v\|_{L^2(\Omega, \gamma)}^2 &= \sum_{k=j+1}^{\infty} |\langle v(\cdot, 0), \psi_k \rangle|^2 \\ &\leq \frac{1}{\lambda_{j+1}^s} \sum_{k=j+1}^{\infty} \lambda_k^s |\langle v(\cdot, 0), \psi_k \rangle|^2 \leq \frac{1}{\lambda_{j+1}^s} \|v\|_{H_{0,L}^1(\mathcal{C}_\Omega, d\gamma(x) \otimes y^a dy)}^2. \end{aligned}$$

Therefore T_j converges to tr_Ω in the operator norm, as $j \rightarrow \infty$, and tr_Ω is compact. \square

Using the previous preliminaries, it is natural to give the following definitions of weak solutions.

Definition 2.5 (Weak solution of (1.6)). *Let $f \in L^2(\Omega, \gamma)$. We say that $w \in H_{0,L}^1(\mathcal{C}_\Omega, d\gamma(x) \otimes y^a dy)$ is a weak solution to the linear Dirichlet-Neumann extension problem (1.6) if*

$$(2.19) \quad \iint_{\mathcal{C}_\Omega} y^a \nabla_{x,y} w \cdot \nabla_{x,y} v d\gamma(x) dy = c_s^{-1} \int_\Omega f(x) v(x, 0) d\gamma(x),$$

for every $v \in H_{0,L}^1(\mathcal{C}_\Omega, d\gamma(x) \otimes y^a dy)$, where $c_s > 0$ is the constant appearing in Theorem 2.2.

Definition 2.6 (Weak solution of (1.1)). *If w is the weak solution to (1.6), its trace $u := w(\cdot, 0) \in \mathcal{H}^s(\Omega, \gamma)$ on Ω will be called a weak solution to (1.1).*

Remark 2.7. *If we assume that f is in the dual space $\mathcal{H}^s(\Omega, \gamma)'$, it is clear that the right hand side in (2.19) must be replaced by the dual product $\langle f, v(\cdot, 0) \rangle$. Then the (unique) solution u to (1.1) will be again the trace over Ω of the unique solution w to the extension problem (1.6).*

The following is just a restatement of Theorem 2.2, see [40, Theorem 1.1] and also [26].

Theorem 2.8 (Extension problem for negative powers). *Given $f \in L^2(\Omega, \gamma)$, let $u \in \mathcal{H}^s(\Omega, \gamma)$ be the unique solution to problem (1.1). The solution w (see (2.13)) to the extension problem (2.14) can be written as*

$$(2.20) \quad \begin{aligned} w(x, y) &= \frac{2^{1-s}}{\Gamma(s)} \sum_{k=1}^{\infty} (\lambda_k^{1/2} y)^s \mathcal{K}_s(\lambda_k^{1/2} y) \frac{\langle f, \psi_k \rangle_{L^2(\Omega, \gamma)}}{\lambda_k^s} \psi_k(x) \\ &= \frac{1}{\Gamma(s)} \int_0^{\infty} e^{-y^2/(4t)} e^{-t\mathcal{L}_{\Omega}} f(x) \frac{dt}{t^{1-s}}. \end{aligned}$$

In particular, this is the unique weak solution to (1.6) and

$$(2.21) \quad w(x, 0) = u(x) = \mathcal{L}^{-s} f(x) = \frac{1}{\Gamma(s)} \int_0^{\infty} e^{-t\mathcal{L}_{\Omega}} f(x) \frac{dt}{t^{1-s}}.$$

The domain $\mathcal{H}^s(\Omega, \gamma)$ of the fractional nonlocal operator \mathcal{L}^s can be characterized as a suitable interpolation space between two Hilbert spaces. Indeed, using the abstract discrete version of the J -Theorem (see for example the Appendix in [12]), it is straightforward to prove that

$$(2.22) \quad \mathcal{H}^s(\Omega, \gamma) = [H_0^1(\Omega, \gamma), L^2(\Omega, \gamma)]_{1-s},$$

where the space in the right hand side of (2.22) is the real interpolation space between $H_0^1(\Omega, \gamma)$ and $L^2(\Omega, \gamma)$. Then $\mathcal{H}^{1/2}(\Omega, \gamma)$ may be seen as the equivalent of the Lions–Magenes space $H_{00}^{1/2}(\Omega)$ in the Gaussian setting.

2.3. Gaussian rearrangements. We give the notion of rearrangement with respect to the Gaussian measure. For extra details, we refer the interested reader to the classical monographs [8] and [19]. If u is a measurable function in Ω , we denote by

- u^{\otimes} the one dimensional decreasing rearrangement of u with respect to the Gaussian measure (also called *one dimensional Gaussian rearrangement of u*):

$$u^{\otimes}(r) = \inf\{t \geq 0 : \gamma_u(t) \leq r\}, \quad r \in (0, \gamma(\Omega)],$$

where $\gamma_u(t) = \gamma(\{x \in \Omega : |u(x)| > t\})$ is the distribution function of u ;

- u^{\star} the n -dimensional rearrangement of u with respect the Gaussian measure:

$$u^{\star}(x) = u^{\otimes}(\Phi(x_1)), \quad x \in \Omega^{\star},$$

where $\Omega^{\star} = \{x = (x_1, \dots, x_n) \in \mathbb{R}^n : x_1 > \lambda\}$ is the half-space such that $\gamma(\Omega^{\star}) = \gamma(\Omega)$ and Φ is given by (2.3).

By definition, u^{\star} is a function which depends only on the first variable x_1 , it is increasing and its level sets are half-spaces. Moreover, u , u^{\otimes} and u^{\star} have the same distribution function. This implies that the Gaussian L^p norm is invariant under these rearrangements:

$$\|u\|_{L^p(\Omega, \gamma)} = \|u^{\otimes}\|_{L^p(0, \gamma(\Omega))} = \|u^{\star}\|_{L^p(\Omega^{\star}, \gamma)}, \quad \text{for any } 1 \leq p \leq \infty.$$

If u is defined on a half-space and $u = u^{\star}$ we sometimes say that u is *rearranged*. Furthermore, if u and v are measurable functions then the following Hardy–Littlewood inequality holds:

$$(2.23) \quad \int_{\Omega} |u(x)v(x)| d\gamma(x) \leq \int_{\Omega^{\star}} u^{\star}(x)v^{\star}(x) d\gamma(x) = \int_0^{\gamma(\Omega)} u^{\otimes}(r)v^{\otimes}(r) dr.$$

If u is defined on Ω , v on Ω^\star and the following estimate holds

$$(2.24) \quad \int_0^{\gamma(\Omega)} u^\otimes(r) dr \leq \int_0^{\gamma(\Omega)} v^\otimes(r) dr,$$

the same inequality is called *mass concentration inequality* (or *comparison of mass concentration*). If $v = v^\star$ and (2.24) occurs, we also say that u^\star is *less concentrated* than v and we write $u^\star \prec v$. Moreover, (2.24) implies that (see for instance [18])

$$\|u\|_{L^p(\Omega, \gamma)} \leq \|v\|_{L^p(\Omega^\star, \gamma)}, \quad \text{for all } 1 \leq p \leq \infty.$$

We will often deal with two-variable functions

$$(2.25) \quad w : (x, y) \in \mathcal{C}_\Omega = \Omega \times (0, \infty) \rightarrow w(x, y) \in \mathbb{R},$$

which are measurable with respect to x . In such a case it will be convenient to consider the so-called *Gaussian Steiner symmetrization* of \mathcal{C}_Ω with respect to the variable x , namely, the set \mathcal{C}_Ω^\star as defined in (1.9). In addition (see for instance [17, 22]) we will denote by $\gamma_w(t, y)$ and $w^\otimes(r, y)$ the distribution function and the one dimensional Gaussian decreasing rearrangements of (2.25), with respect to x , for each y fixed. We will also define the function

$$w^\star(x, y) = w^\otimes(\Phi(x_1), y),$$

which is called the *Gaussian Steiner symmetrization* of w , with respect to x , that is, with respect to the line $x = 0$. Clearly, for any fixed y , $w^\star(\cdot, y)$ is an increasing function depending only on x_1 .

Now we recall a result that we will use in the proof of our main comparison result in Section 3.

Proposition 2.9 (See [17, p. 255]). *Consider the Cauchy–Dirichlet problem (2.8) with $\Omega = \Omega^\star$. If $f(x) = f^\star(x)$ for a.e. $x \in \Omega^\star$ and $f^\star \in L^2(\Omega^\star, \gamma)$, then the solution η to (2.8) is such that $\eta(x, t) = \eta^\star(x, t)$, for a.e. $x \in \Omega^\star$ and for all $t \geq 0$.*

2.4. The second order derivation formula. It will be essential for us to be able to differentiate with respect to the extra variable y under the integral symbol in the expression

$$\int_{\{w(x, y) > w^\otimes(r, y)\}} \frac{\partial w}{\partial y}(x, y) d\gamma(x).$$

Equivalently, we need to derive the Gaussian version of the first and second order differentiation formulas established for the Lebesgue measure in [4, 7, 25, 32]. The first order differentiation formula can be stated as follows:

Proposition 2.10 (See [17], also [37]). *If $w \in H^1(0, T; L^2(\Omega, \gamma))$ is a nonnegative function, for some $T > 0$, then $w^\otimes \in H^1(0, T; L^2(0, \gamma(\Omega)))$. In addition, if $\gamma(\{w(x, t) = w^\otimes(r, t)\}) = 0$ for a.e. $(r, t) \in (0, \gamma(\Omega)) \times (0, T)$, then the following derivation formula holds*

$$(2.26) \quad \int_{w(x, y) > w^\otimes(r, y)} \frac{\partial}{\partial y} w(x, y) d\gamma(x) = \int_0^r w^\otimes(\sigma, y) d\sigma.$$

In order to prove our novel second order derivation formula, we need the following version of the coarea formula (see [23] and [27, Theorem 11]).

Proposition 2.11. *If $u \in W_{\text{loc}}^{1,p}(\mathbb{R}^n)$, with $p > 1$, and $\psi : \mathbb{R}^n \rightarrow \mathbb{R}$ is a nonnegative measurable function, then there exists a representative of u , denoted again by u , such that*

$$(2.27) \quad \int_{\mathbb{R}^n} \psi(x) |\nabla_x u| dx = \int_{-\infty}^{\infty} \left(\int_{u(x)=\tau} \psi(x) d\mathcal{H}^{n-1}(x) \right) d\tau.$$

Now we present our new Gaussian derivation formulas, which are a nonstandard adaptation of the derivation formula exhibited in [25].

Theorem 2.12. *Let $0 < \varepsilon < T < \infty$. Consider a nonnegative function*

$$w = w(x, y) \in H_{0,L}^1(\mathcal{C}_\Omega, d\gamma(x) \otimes y^\alpha dy) \cap C^1(\Omega \times (\varepsilon, T)).$$

Suppose also that w is $C^{1,\alpha}$ with respect to $y \in (\varepsilon, T)$, for some $0 < \alpha \leq 1$, uniformly with respect to $x \in \Omega$. Moreover, assume that $f(x, y)$ is a continuous function on the cylinder \mathcal{C}_Ω such that $f \in H^1(\mathcal{C}_\Omega, d\gamma(x) \otimes y^\alpha dy)$ and the function $f(x, y)\varphi(x)$ is Lipschitz with respect to $y \in (\varepsilon, T)$, uniformly with respect to $x \in \Omega$. Furthermore, suppose that

$$(2.28) \quad \gamma\left(\left\{x \in \Omega : |\nabla_x w| = 0, w(x, y) \in (0, \sup_x w(x, y))\right\}\right) = 0, \quad \text{for all } y \in (\varepsilon, T),$$

and set

$$H(t, y) := \int_{w(x, y) > t} f(x, y) d\gamma(x),$$

for $t \in [0, \infty)$ and $y \in (\varepsilon, T)$. The following statements hold true.

(i) For any fixed $y \in (\varepsilon, T)$, $H(t, y)$ is differentiable with respect to t for a.e. $t \geq 0$ and

$$(2.29) \quad \frac{\partial}{\partial t} H(t, y) = - \int_{w(x, y) = t} \frac{f(x, y)}{|\nabla_x w|} \varphi(x) d\mathcal{H}^{n-1}(x).$$

(ii) For any fixed $t \geq 0$, $H(t, y)$ is differentiable with respect to y and, for a.e. $y \in (\varepsilon, T)$,

$$(2.30) \quad \frac{\partial}{\partial y} H(t, y) = \int_{w(x, y) > t} \frac{\partial}{\partial y} f(x, y) d\gamma(x) + \int_{w(x, y) = t} \frac{\partial}{\partial y} w(x, y) \frac{f(x, y)}{|\nabla_x w|} \varphi(x) d\mathcal{H}^{n-1}(x).$$

Proof. Let us first prove (i). By the extension theorem (see for instance [24]) we can extend $w(\cdot, y)$ as a function in $H^1(\mathbb{R}^n)$, for a.e. $y > 0$. Condition (2.28) allows us to choose $\psi(x) = \frac{f(x, y)}{|\nabla_x w|} \varphi(x) \chi_{\{w(x, y) > t\}}(x)$ and $u(x) = w(x, y)$ in the coarea formula (2.27) to get

$$\int_{w(x, y) > t} f(x, y) d\gamma(x) = \int_t^\infty \left(\int_{w(x, y) = \tau} \frac{f(x, y)}{|\nabla_x w|} \varphi(x) d\mathcal{H}^{n-1}(x) \right) d\tau,$$

for a.e. $t \geq 0$. Thus (2.29) follows.

Next we prove (ii). We observe that

$$H(t, y) - H(t, \bar{y}) = \triangle_1 + \triangle_2 + \triangle_3,$$

where

$$\triangle_1 = \int_{w(x, \bar{y}) > t} [f(x, y) - f(x, \bar{y})] d\gamma(x), \quad \triangle_2 = \int_{w(x, y) > t \geq w(x, \bar{y})} f(x, y) d\gamma(x),$$

and

$$\triangle_3 = - \int_{w(x, \bar{y}) > t \geq w(x, y)} f(x, y) d\gamma(x).$$

Since $f(x, y)\varphi(x)$ is Lipschitz with respect to y , uniformly in x , by Lebesgue's dominated convergence theorem we easily infer that

$$(2.31) \quad \lim_{y \rightarrow \bar{y}} \frac{\triangle_1}{y - \bar{y}} = \int_{w(x, \bar{y}) > t} \frac{\partial f}{\partial y}(x, \bar{y}) d\gamma(x),$$

for a.e. t and a.e. $\bar{y} \in (\varepsilon, T)$. Let us next consider $\frac{\triangle_2}{y - \bar{y}}$. We have

$$(2.32) \quad \frac{\triangle_2}{y - \bar{y}} = \frac{1}{y - \bar{y}} \int_{D_1} f(x, y) d\gamma(x) + \frac{1}{y - \bar{y}} \int_{D_2} f(x, y) d\gamma(x),$$

where

$$D_1 = \left\{ x \in \Omega : w(x, y) > t \geq w(x, \bar{y}), \frac{\partial w}{\partial y}(x, \bar{y}) = 0 \right\},$$

and

$$D_2 = \left\{ x \in \Omega : w(x, y) > t \geq w(x, \bar{y}), \frac{\partial w}{\partial y}(x, \bar{y}) \neq 0 \right\}.$$

We claim that

$$(2.33) \quad \lim_{y \rightarrow \bar{y}} \frac{1}{y - \bar{y}} \int_{D_1} f(x, y) d\gamma(x) = 0, \quad \text{for a.e. } t \geq 0.$$

Since $w(x, y) \in C^{1,\alpha}$ with respect to $y \in (\varepsilon, T)$, uniformly in $x \in \Omega$, we have

$$(2.34) \quad \left| \frac{\partial w}{\partial y}(x, y) - \frac{\partial w}{\partial y}(x, \bar{y}) \right| \leq c|y - \bar{y}|^\alpha, \quad \text{for every } x \in \Omega,$$

for a constant $c > 0$ independent on x, y and \bar{y} . Since for any $x \in D_1$ we have $\frac{\partial}{\partial y}w(x, \bar{y}) = 0$, by (2.34) we easily find the uniform estimate

$$|w(x, y) - w(x, \bar{y})| \leq \int_{\bar{y}}^y \left| \frac{\partial}{\partial z}w(x, z) \right| dz \leq c|y - \bar{y}|^{\alpha+1}, \quad \text{for all } x \in D_1,$$

which yields

$$(2.35) \quad \left| \frac{1}{y - \bar{y}} \int_{D_1} f(x, y) d\gamma(x) \right| \leq \frac{1}{|y - \bar{y}|} \int_{\{x: t - c|y - \bar{y}|^{\alpha+1} < w(x, \bar{y}) \leq t\}} |f(x, y)| d\gamma(x).$$

Let us set

$$\Psi(t) := \int_{w(x, \bar{y}) > t} \max_{y \in [\varepsilon, T]} |f(x, y)| d\gamma(x).$$

Since $f \in L^2(\mathcal{C}_\Omega, d\gamma(x) \otimes y^\alpha dy)$ and f is continuous, by Fubini's theorem we have that

$$\int_{\Omega} |f(x, y)| d\gamma(x) < \infty,$$

for a.e. $y > 0$, and $\Psi(t) < \infty$, for all $t \geq 0$. Then (2.35) implies

$$\left| \frac{1}{y - \bar{y}} \int_{D_1} f(x, y) d\gamma(x) \right| \leq c|y - \bar{y}|^\alpha \frac{\Psi(t - c|y - \bar{y}|^{\alpha+1}) - \Psi(t)}{c|y - \bar{y}|^{\alpha+1}}.$$

Since the function Ψ is monotone, it is also differentiable almost everywhere and then (2.33) holds. Now let us evaluate the second term in (2.32). First we consider the case $y > \bar{y}$. For y sufficiently close to \bar{y} , we have

$$\frac{1}{y - \bar{y}} \int_{D_2} f(x, y) d\gamma(x) = \frac{1}{y - \bar{y}} \int_{D_3} f(x, y) d\gamma(x),$$

where

$$D_3 = \left\{ x \in \Omega : w(x, y) > t \geq w(x, \bar{y}), \frac{\partial w}{\partial y}(x, \bar{y}) > 0 \right\}.$$

Let us set

$$\Gamma_t = \{x \in \Omega : w(x, \bar{y}) = t\} \cap \left\{ x \in \Omega : \frac{\partial w}{\partial y}(x, \bar{y}) > 0 \right\}.$$

In a neighborhood $B_r(\bar{x}, \bar{y}, t)$ of a point $(\bar{x}, \bar{y}, t) \in \mathbb{R}^{n+2}$ with $\bar{x} \in \Gamma_t$, the equality $w(x, y) = t$ implicitly defines a function $y = v(x, t)$ such that $\bar{y} = v(\bar{x}, t)$ and $w(x, v(x, t)) = t$. Moreover for y sufficiently close to \bar{y} we have

$$D_3 \cap B_r(\bar{x}, \bar{y}, t) = \{x \in B_r(\bar{x}, \bar{y}, t) : \bar{y} < v(x, t) < y\}.$$

Observe that the implicit function theorem gives $|\nabla_x v(x, t)| = |\nabla_x w(x, \bar{y})| / \frac{\partial w}{\partial y}(x, \bar{y})$. Then using the coarea formula (2.27) we have

$$(2.36) \quad \begin{aligned} \lim_{y \rightarrow \bar{y}^+} \frac{1}{y - \bar{y}} \int_{D_3 \cap B_r(\bar{x}, \bar{y}, t)} f(x, y) d\gamma(x) &= \lim_{y \rightarrow \bar{y}^+} \frac{1}{y - \bar{y}} \int_{\bar{y}}^y \int_{v(x, t)=s} \frac{f(x, y) \varphi(x)}{|\nabla_x v|} d\mathcal{H}^{n-1}(x) ds \\ &= \int_{\{x \in B_r(\bar{x}, \bar{y}, t) : v(x, t) = \bar{y}\}} \frac{f(x, \bar{y})}{|\nabla_x v|} \varphi(x) d\mathcal{H}^{n-1}(x) \\ &= \int_{\{x \in B_r(\bar{x}, \bar{y}, t) : w(x, \bar{y}) = t\}} \frac{\partial w}{\partial y}(x, \bar{y}) \frac{f(x, \bar{y})}{|\nabla_x w|} \varphi(x) d\mathcal{H}^{n-1}(x). \end{aligned}$$

By (2.33) and (2.36) it follows that

$$(2.37) \quad \lim_{y \rightarrow \bar{y}^+} \frac{\Delta_2}{y - \bar{y}} = \int_{\{x: w(x, \bar{y}) = t, \frac{\partial w}{\partial y}(x, \bar{y}) > 0\}} \frac{\partial}{\partial y} w(x, \bar{y}) \frac{f(x, \bar{y})}{|\nabla_x w|} \varphi(x) d\mathcal{H}^{n-1}(x).$$

By analogous arguments we obtain

$$(2.38) \quad \lim_{y \rightarrow \bar{y}} \frac{\Delta_2}{y - \bar{y}} = \int_{\{x: w(x, \bar{y})=t, \frac{\partial w}{\partial y}(x, \bar{y}) < 0\}} \frac{\partial w}{\partial y}(x, \bar{y}) \frac{f(x, \bar{y})}{|\nabla_x w|} \varphi(x) d\mathcal{H}^{n-1}(x).$$

In the same way we can prove the analogue of (2.37) and (2.38) with Δ_2 replaced by Δ_3 . Then

$$(2.39) \quad \lim_{y \rightarrow \bar{y}} \frac{\Delta_2 + \Delta_3}{y - \bar{y}} = \int_{\{x: w(x, \bar{y})=t\}} \frac{\partial w}{\partial y}(x, \bar{y}) \frac{f(x, \bar{y})}{|\nabla_x w|} \varphi(x) d\mathcal{H}^{n-1}(x).$$

Putting together (2.31) and (2.39) we obtain assertion (ii). \square

By recalling that the rearrangement w^\circledast of a function w is the generalized inverse function of the distribution function γ_w , and applying the chain rule formula, we can prove our novel derivation formula.

Corollary 2.13 (Gaussian second order derivation formula). *Under the assumptions of Theorem 2.12, for a.e. $y \in (\varepsilon, T)$ the following derivation formula holds:*

$$(2.40) \quad \begin{aligned} \frac{\partial}{\partial y} \int_{w(x, y) > w^\circledast(r, y)} f(x, y) d\gamma(x) &= \int_{w(x, y) > w^\circledast(r, y)} \frac{\partial}{\partial y} f(x, y) d\gamma(x) \\ &- \int_{w(x, y) = w^\circledast(r, y)} \frac{f(x, y)}{|\nabla_x w|} \left[\frac{\int_{w(x, y) = w^\circledast(r, y)} \frac{\frac{\partial}{\partial y} w(x, y)}{|\nabla_x w|} \varphi(x) d\mathcal{H}^{n-1}(x)}{\int_{w(x, y) = w^\circledast(r, y)} \frac{1}{|\nabla_x w|} \varphi(x) d\mathcal{H}^{n-1}(x)} - \frac{\partial}{\partial y} w(x, y) \right] \varphi(x) d\mathcal{H}^{n-1}(x). \end{aligned}$$

In particular, if $w(x, y)$ is C^1 and the functions $w(x, y)\varphi(x)$, $\frac{\partial}{\partial y} w(x, y)\varphi(x)$ are Lipschitz in $y \in (\varepsilon, T)$, uniformly with respect to $x \in \Omega$, we have

$$(2.41) \quad \begin{aligned} &\int_{w(x, y) > w^\circledast(r, y)} \frac{\partial^2}{\partial y^2} w(x, y) d\gamma(x) \\ &= \frac{\partial^2}{\partial y^2} \int_0^r w^\circledast(\sigma, y) d\sigma - \int_{w(x, y) = w^\circledast(r, y)} \frac{\left(\frac{\partial}{\partial y} w(x, y)\right)^2}{|\nabla_x w|} \varphi(x) d\mathcal{H}^{n-1}(x) \\ &\quad + \left(\int_{w(x, y) = w^\circledast(r, y)} \frac{\frac{\partial}{\partial y} w(x, y)}{|\nabla_x w|} \varphi(x) d\mathcal{H}^{n-1}(x) \right)^2 \left(\int_{w(x, y) = w^\circledast(r, y)} \frac{\varphi(x)}{|\nabla_x w|} d\mathcal{H}^{n-1}(x) \right)^{-1}. \end{aligned}$$

Proof. In order to prove (2.40) we need to evaluate the y -derivative of $H(t, y)$ when $t = w^\circledast(r, y)$. By a rearrangement property (see for example [8]) we have

$$(2.42) \quad \int_{w(x, y) > w^\circledast(r, y)} w(x, y) d\gamma(x) = \int_0^s w^\circledast(\sigma, y) d\sigma.$$

Observe that by applying (2.27) it is not difficult to prove that

$$(2.43) \quad -\frac{\partial w^\circledast}{\partial r} = \left(\int_{w(x, y) = w^\circledast(r, y)} \frac{\varphi(x)}{|\nabla_x w|} d\mathcal{H}^{n-1}(x) \right)^{-1}.$$

Now using (2.27), (2.43), (2.26) and the chain rule,

$$\begin{aligned}
(2.44) \quad \frac{\partial}{\partial y} w^\otimes(r, y) &= \frac{\partial}{\partial y} \left(\frac{\partial}{\partial r} \int_0^r w^\otimes(\tau, y) d\tau \right) \\
&= \frac{\partial}{\partial r} \left(\frac{\partial}{\partial y} \int_0^r w^\otimes(\tau, y) d\tau \right) = \frac{\partial}{\partial r} \int_{w(x, y) > w^\otimes(r, y)} \frac{\partial w}{\partial y} d\gamma(x) \\
&= \frac{\partial}{\partial r} \int_{w=w^\otimes(r, y)}^\infty d\tau \int_{w(x, y)=\tau} \frac{\frac{\partial}{\partial y} w(x, y)}{|\nabla_x w|} \varphi(x) d\mathcal{H}^{n-1}(x) \\
&= -\frac{\partial w^\otimes}{\partial r} \int_{w(x, y)=w^\otimes(r, y)} \frac{\frac{\partial}{\partial y} w(x, y)}{|\nabla_x w|} \varphi(x) d\mathcal{H}^{n-1}(x) \\
&= \frac{\int_{w(x, y)=w^\otimes(r, y)} \frac{\frac{\partial}{\partial y} w(x, y)}{|\nabla_x w|} \varphi(x) d\mathcal{H}^{n-1}(x)}{\int_{w(x, y)=w^\otimes(r, y)} \frac{1}{|\nabla_x w|} \varphi(x) d\mathcal{H}^{n-1}(x)}.
\end{aligned}$$

By (2.44), (2.29) and (2.30) we obtain

$$\begin{aligned}
(2.45) \quad \frac{\partial}{\partial y} H(w^\otimes(r, y), y) &= \frac{\partial}{\partial t} H(t, y) \Big|_{t=w^\otimes(r, y)} \frac{\partial}{\partial y} w^\otimes(r, y) + H_y(w^\otimes(r, y), y) \\
&= - \int_{w(x, y)=w^\otimes(r, y)} \frac{f(x, y)}{|\nabla_x w|} \varphi(x) d\mathcal{H}^{n-1}(x) \times \frac{\int_{w(x, y)=w^\otimes(r, y)} \frac{\frac{\partial}{\partial y} w(x, y)}{|\nabla_x w|} \varphi(x) d\mathcal{H}^{n-1}(x)}{\int_{w(x, y)=w^\otimes(r, y)} \frac{1}{|\nabla_x w|} \varphi(x) d\mathcal{H}^{n-1}(x)} \\
&\quad + \int_{w(x, y) > w^\otimes(r, y)} \frac{\partial}{\partial y} f(x, y) d\gamma(x) + \int_{w(x, y)=w^\otimes(r, y)} \frac{\partial}{\partial y} w(x, y) \frac{f(x, y)}{|\nabla_x w|} \varphi(x) d\mathcal{H}^{n-1}(x),
\end{aligned}$$

which is (2.40). Now we are in position to prove (2.41). Indeed, by applying (2.45) with $f(x, y) = w_y(x, y)$ and (2.26), we finally get

$$\begin{aligned}
\frac{\partial^2}{\partial y^2} \int_0^r w^\otimes(\sigma, y) d\sigma &= \frac{\partial}{\partial y} \int_{w(x, y) > w^\otimes(r, y)} \frac{\partial}{\partial y} w(x, y) d\gamma(x) \\
&= \int_{w(x, y) > w^\otimes(r, y)} \frac{\partial^2}{\partial y^2} w(x, y) d\gamma(x) \\
&\quad + \int_{w(x, y)=w^\otimes(r, y)} \frac{\left(\frac{\partial}{\partial y} w(x, y) \right)^2}{|\nabla_x w|} \varphi(x) d\mathcal{H}^{n-1}(x) \\
&\quad - \frac{\left(\int_{w(x, y)=w^\otimes(r, y)} \frac{\frac{\partial}{\partial y} w(x, y)}{|\nabla_x w|} \varphi(x) d\mathcal{H}^{n-1}(x) \right)^2}{\left(\int_{w(x, y)=w^\otimes(r, y)} \frac{1}{|\nabla_x w|} \varphi(x) d\mathcal{H}^{n-1}(x) \right)}.
\end{aligned}$$

□

Remark 2.14. *The sum of the last two terms to the right-hand side of (2.41) is nonpositive, see [3, Remark 2.8].*

The following Lemma shows that we can actually apply the second order derivation formula (2.41) to the solution w to the extension problem (1.6), namely, when $w = \mathcal{P}_y^s u$ is the extension of the solution $u \in \mathcal{H}^s(\Omega, \gamma)$ to the linear problem (1.1).

Lemma 2.15. *If $f \in L^2(\Omega, \gamma)$ then the second order derivation formula (2.41) can be applied to the solution w to problem (1.6).*

Proof. Since $w \in C^\infty(\mathcal{C}_\Omega)$, by classical results on solutions of elliptic equations with analytic coefficients (see for instance [28]), w is analytic. Hence condition (2.28) holds. Next we have to show that the functions $w(x, y)\varphi(x)$ and $\partial_y w(x, y)\varphi(x)$ are Lipschitz in $y \in (\varepsilon, T)$, uniformly with respect to $x \in \Omega$. This follows because it is known that the solution to the extension problem has the regularity $w \in C^\infty((0, \infty); \mathcal{H}^s(\Omega, \gamma))$, see [26, 40]. For the sake of completeness, we also give a direct proof of this regularity result. By Theorem 2.8 and using the well known identity $\frac{d}{dt}(t^\nu \mathcal{K}_\nu(t)) = -t^\nu \mathcal{K}_{\nu-1}(t)$, for $\nu \in \mathbb{R}$, it follows that

$$\partial_y w = -C_s \sum_{k=1}^{\infty} (\lambda_k^{1/2} y)^s \mathcal{K}_{s-1}(\lambda_k^{1/2} y) \frac{\langle f, \psi_k \rangle_{L^2(\Omega, \gamma)}}{\lambda_k^{s-1/2}} \psi_k(x)$$

and

$$\partial_{yy} w = -C_s \sum_{k=1}^{\infty} [(\lambda_k^{1/2} y)^{s-1} \mathcal{K}_{s-1}(\lambda_k^{1/2} y) - (\lambda_k^{1/2} y)^{s-1} \mathcal{K}_{s-2}(\lambda_k^{1/2} y)] \frac{\langle f, \psi_k \rangle_{L^2(\Omega, \gamma)}}{\lambda_k^{s-1}} \psi_k(x).$$

Then, as $\mathcal{K}_\nu(t) \sim \sqrt{\frac{\pi}{2t}} e^{-t}$, as $t \rightarrow \infty$, and $\mathcal{K}_\nu(t) \sim C_\nu t^{-\nu}$, as $t \rightarrow 0$, we get

$$\begin{aligned} \int_0^\infty y^a \int_\Omega |\partial_y w|^2 d\gamma(x) dy &= C_s \sum_{k=1}^{\infty} \left[\int_0^\infty y^a |(\lambda_k^{1/2} y)^s \mathcal{K}_{s-1}(\lambda_k^{1/2} y)|^2 dy \right] \frac{|\langle f, \psi_k \rangle_{L^2(\Omega, \gamma)}|^2}{\lambda_k^{2s-1}} \\ &= C_s \sum_{k=1}^{\infty} \left[\int_0^\infty r |\mathcal{K}_{s-1}(r)|^2 dr \right] \frac{|\langle f, \psi_k \rangle_{L^2(\Omega, \gamma)}|^2}{\lambda_k^s} \leq \frac{C_s}{\lambda_1^s} \|f\|_{L^2(\Omega, \gamma)}^2, \end{aligned}$$

and $w(x, y)\varphi(x)$ is Lipschitz with respect to $y \in (0, \infty)$, uniformly in x . On the other hand,

$$\begin{aligned} \int_\varepsilon^\infty y^a \int_\Omega |\partial_{yy} w|^2 d\gamma(x) dy &\leq C_{s, \varepsilon} \sum_{k=1}^{\infty} \int_\varepsilon^\infty y^a |(\lambda_k^{1/2} y)^{s-1} [(\lambda_k^{1/2} y)^{-1/2} e^{-\lambda_k^{1/2} y} (1+y)]|^2 dy \frac{|\langle f, \psi_k \rangle_{L^2(\Omega, \gamma)}|^2}{\lambda_k^{2s-2}} \\ &\leq C_{s, \varepsilon} \sum_{k=1}^{\infty} \int_\varepsilon^\infty y^{-2} e^{-2\lambda_k^{1/2} y} (1+y)^2 dy \frac{|\langle f, \psi_k \rangle_{L^2(\Omega, \gamma)}|^2}{\lambda_k^{s-1/2}} \\ &\leq C_{s, \varepsilon} \sum_{k=1}^{\infty} \int_\varepsilon^\infty e^{-2\lambda_k^{1/2} y} dy \frac{|\langle f, \psi_k \rangle_{L^2(\Omega, \gamma)}|^2}{\lambda_k^{s-1/2}} \\ &= C_{s, \varepsilon} \sum_{k=1}^{\infty} \frac{e^{-2\varepsilon \lambda_k^{1/2}}}{\lambda_k^s} |\langle f, \psi_k \rangle_{L^2(\Omega, \gamma)}|^2 \leq \frac{C_{s, \varepsilon}}{\lambda_1^s} \|f\|_{L^2(\Omega, \gamma)}^2. \end{aligned}$$

Hence $\partial_y w(x, y)\varphi(x)$ is Lipschitz with respect to $y \in (\varepsilon, \infty)$, uniformly in $x \in \Omega$. \square

3. THE COMPARISON RESULT

With the previous results at hand, we are now in position to prove the main result of the paper.

Theorem 3.1 (Comparison result). *Let Ω be an open subset of \mathbb{R}^n with $\gamma(\Omega) < 1$. Let u and ψ be the weak solutions to (1.1) and (1.5), respectively, with $f \in L^2(\Omega, \gamma)$. Then*

$$(3.1) \quad \int_0^r u^\otimes(\sigma) d\sigma \leq \int_0^r \psi^\otimes(\sigma) d\sigma, \quad \text{for all } 0 \leq r \leq \gamma(\Omega),$$

that is,

$$u^\star \prec \psi.$$

Proof. By making the change of variables $y = (2s)z^{1/(2s)}$ (see [14]), we can write the extension problems (1.6) and (1.8) as

$$(3.2) \quad \begin{cases} -\mathcal{L}w + z^{2-1/s}w_{zz} = 0, & \text{in } \mathcal{C}_\Omega, \\ w = 0, & \text{on } \partial_L \mathcal{C}_\Omega, \\ -\lim_{z \rightarrow 0^+} w_z = d_s f, & \text{on } \Omega. \end{cases}$$

and

$$(3.3) \quad \begin{cases} -\mathcal{L}v + z^{2-1/s}v_{zz} = 0, & \text{in } \mathcal{C}_\Omega^\star, \\ v = 0, & \text{on } \partial_L \mathcal{C}_\Omega^\star, \\ -\lim_{z \rightarrow 0^+} v_z = d_s f^\star, & \text{on } \Omega^\star, \end{cases}$$

for some explicit constant $d_s > 0$, respectively. Now, since u is the trace on Ω of the solution w to (3.2) and ψ is the trace on Ω^\star of the solution v to (3.3), the result will immediately follow once we prove the concentration comparison inequality

$$(3.4) \quad \int_0^r w^\oplus(\sigma, z) d\sigma \leq \int_0^r v^\oplus(\sigma, z) d\sigma, \quad \text{for all } 0 \leq r \leq \gamma(\Omega), \text{ for any fixed } z \geq 0.$$

We recall that w is smooth for any $z > 0$. For a fixed $z > 0$ and $t > 0$, let

$$\varsigma_h^z(x) := \begin{cases} \text{sign } w(x, z), & \text{if } |w(x, z)| \geq t + h, \\ \frac{|w(x, z)| - t}{h} \text{sign } w, & \text{if } t < |w(x, z)| < t + h, \\ 0, & \text{otherwise.} \end{cases}$$

By multiplying the first equation in (3.2) by $\varsigma_h^z(x)$ and integrating over Ω with respect to the Gaussian measure, we obtain

$$\frac{1}{h} \int_{t < |w| < t+h} |\nabla_x w|^2 d\gamma - \frac{z^{2-1/s}}{h} \int_{|w| > t+h} \frac{\partial^2 w}{\partial z^2} d\gamma - \frac{z^{2-1/s}}{h} \int_{t < |w| < t+h} \frac{\partial^2 w}{\partial z^2} (|w| - t) \text{sign } w d\gamma = 0.$$

Letting $h \rightarrow 0$ we obtain

$$(3.5) \quad -\frac{\partial}{\partial t} \int_{|w| > t} |\nabla_x w|^2 d\gamma(x) - z^{2-1/s} \int_{|w| > t} \frac{\partial^2 w}{\partial z^2} d\gamma(x) = 0.$$

On the other hand, the coarea formula (2.27) and the isoperimetric inequality with respect to the Gaussian measure (2.2) give

$$(3.6) \quad -\frac{\partial}{\partial t} \int_{|w| > t} |\nabla_x w| d\gamma(x) \geq \int_{\partial\{|w| > t\}^\star} \varphi(x) d\mathcal{H}^{n-1}(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{[\Phi^{-1}(\gamma_w(t))]^2}{2}\right),$$

where $\{|w| > t\}^\star$ is the half-space having Gauss measure $\gamma_w(t)$. By Hölder's inequality,

$$\frac{1}{h} \int_{t+h > |w| > t} |\nabla_x w| d\gamma(x) \leq \left(\frac{1}{h} \int_{t+h > |w| > t} |\nabla_x w|^2 d\gamma(x) \right)^{1/2} \left(\frac{1}{h} \int_{t+h > |w| > t} d\gamma(x) \right)^{1/2},$$

for any $h > 0$. Hence, by taking $h \rightarrow 0$,

$$-\frac{\partial}{\partial t} \int_{|w| > t} |\nabla_x w| d\gamma(x) \leq \left(-\frac{\partial}{\partial t} \int_{|w| > t} |\nabla_x w|^2 d\gamma(x) \right)^{1/2} \left(-\frac{\partial}{\partial t} \int_{|w| > t} d\gamma(x) \right)^{1/2}.$$

Then (3.6) yields

$$(3.7) \quad -\frac{\partial}{\partial t} \int_{|w| > t} |\nabla_x w|^2 d\gamma(x) \geq \frac{1}{2\pi} (-\gamma'_w(t))^{-1} \exp\left(-[\Phi^{-1}(\gamma_w(t))]^2\right).$$

By plugging (3.7) into (3.5) we have

$$-z^{2-1/s} \int_{|w|>t} \frac{\partial^2 w}{\partial z^2} d\gamma(x) - \frac{1}{2\pi} (\gamma'_w(t))^{-1} \exp\left(-[\Phi^{-1}(\gamma_w(t))]^2\right) \leq 0.$$

Now we set

$$W(r, y) := \int_0^r w^\oplus(\sigma, z) d\sigma.$$

Using Lemma 2.15 and the second order derivation formula (2.41) we find that W verifies the following differential inequality

$$(3.8) \quad -z^{2-1/s} \frac{\partial^2 W}{\partial z^2} - p(r) \frac{\partial^2 W}{\partial r^2} \leq 0$$

for a.e. $(r, z) \in (0, \gamma(\Omega)) \times (0, \infty)$, where $p(r) = \frac{1}{2\pi} \exp(-[\Phi^{-1}(r)]^2)$. Moreover, the first order derivation formula (2.26) implies

$$\frac{\partial W}{\partial z}(r, z) = \frac{\partial}{\partial z} \int_{w(x, z) > w^\oplus(r, z)} w(x, z) d\gamma(x) = \int_{w(x, z) > w^\oplus(r, z)} \frac{\partial}{\partial z} w(x, z) d\gamma(x).$$

Then, by the Hardy–Littlewood inequality (2.23), we easily infer

$$\begin{aligned} \frac{\partial W}{\partial z}(r, 0) &= \int_{w(x, 0) > w^\oplus(r, 0)} \frac{\partial w}{\partial z}(x, 0) d\gamma(x) = -d_s \int_{u(x) > u^\oplus(r)} f(x) d\gamma(x) \\ &\geq -d_s \int_0^r f^\oplus(\sigma) d\sigma, \quad \text{for } r \in (0, \gamma(\Omega)). \end{aligned}$$

Therefore W satisfies the following boundary conditions

$$\begin{aligned} W(0, z) &= 0, \quad z \in [0, \infty), \\ \frac{\partial W}{\partial r}(\gamma(\Omega), z) &= 0, \quad z \in [0, \infty), \\ \frac{\partial W}{\partial z}(r, 0) &\geq -d_s \int_0^r f^\oplus(\sigma) d\sigma, \quad \text{for } r \in (0, \gamma(\Omega)). \end{aligned}$$

Next let us turn our attention to problem (1.8). By Proposition 2.9, it follows that the function $\eta(x, t) := (e^{-t(\mathcal{L}_\Omega^\star)} f^\star)(x)$, is rearranged with respect to x , that is, $\eta(x, t) = \eta^\star(x, t)$. Recall the semigroup formula (2.20):

$$v(x, y) = \frac{1}{\Gamma(s)} \int_0^\infty e^{-y^2/(4t)} \eta(x, t) \frac{dt}{t^{1-s}}.$$

It is then clear that (even after the change of variables $y = (2s)z^{1/(2s)}$) v is rearranged with respect to x as well, namely, $v(x, z) = v^\star(x, z)$. This implies that the level sets of $v(\cdot, z)$ are half-spaces, which gives in turn that all the inequalities involved in the symmetrization arguments for the solution u we performed above become equalities for v . Therefore, if

$$V(r, z) := \int_0^r v^\oplus(\sigma, z) d\sigma,$$

then

$$(3.9) \quad -z^{2-1/s} \frac{\partial^2 V}{\partial z^2} - p(r) \frac{\partial^2 V}{\partial r^2} = 0.$$

Regarding the boundary conditions, we have

$$\begin{aligned} \frac{\partial V}{\partial z}(r, 0) &= -d_s \int_{\psi(x_1) > \psi^\oplus(r)} f^\star(x) d\gamma(x) \\ &= -d_s \int_{\Phi^{-1}(r)}^\infty f^\oplus(\Phi^{-1}(x_1)) d\gamma(x) \\ &= -d_s \int_0^r f^\oplus(\sigma) d\sigma, \quad \text{for } r \in (0, \gamma(\Omega)). \end{aligned}$$

Then V satisfies:

$$\begin{aligned} V(0, z) &= 0, \quad z \in [0, \infty), \\ \frac{\partial V}{\partial r}(\gamma(\Omega), z) &= 0, \quad z \in [0, \infty), \\ \frac{\partial V}{\partial z}(r, 0) &= -d_s \int_0^r f^{\otimes}(\sigma) d\sigma, \quad \text{for } r \in (0, \gamma(\Omega)). \end{aligned}$$

If we put $Z(r, z) := W(r, z) - V(r, z) = \int_0^r [w^{\otimes}(\sigma, z) - v^{\otimes}(\sigma, z)] d\sigma$, then (3.8) and (3.9) imply that Z is a subsolution to

$$(3.10) \quad -z^{2-1/s} \frac{\partial^2 Z}{\partial z^2} - p(r) \frac{\partial^2 Z}{\partial r^2} \leq 0,$$

for a.e. $(r, z) \in (0, \gamma(\Omega)) \times (0, \infty)$, together with the following boundary conditions

$$(3.11) \quad \begin{aligned} Z(0, z) &= 0, \quad z \in [0, \infty), \\ \frac{\partial Z}{\partial r}(\gamma(\Omega), z) &= 0, \quad z \in [0, \infty), \\ \frac{\partial Z}{\partial z}(r, 0) &\geq 0, \quad \text{for } r \in (0, \gamma(\Omega)). \end{aligned}$$

Moreover, since $\|w(\cdot, z)\|_{L^2(\Omega, \gamma)}, \|v(\cdot, z)\|_{L^2(\Omega^*, \gamma)} \rightarrow 0$, as $z \rightarrow \infty$, we have $Z(r, z) \rightarrow 0$, as $z \rightarrow \infty$, uniformly in r . Now we claim that $Z \leq 0$ in $[0, \gamma(\Omega)] \times [0, \infty)$. Indeed, observe that (3.10) can be rewritten as

$$-p(r)^{-1} \frac{\partial^2 Z}{\partial z^2} - z^{-2+1/s} \frac{\partial^2 Z}{\partial r^2} \leq 0.$$

Therefore, by multiplying both sides by Z_+ , the positive part of Z , and integrating by parts over the strip $(0, \gamma(\Omega)) \times (0, \infty)$, the boundary conditions (3.11) and the fact that $Z(r, z) \rightarrow 0$ as $z \rightarrow \infty$ imply

$$\begin{aligned} \int_0^{\gamma(\Omega)} p(r)^{-1} \frac{\partial Z}{\partial z}(r, 0) Z_+(r, 0) dr + \int_0^\infty \int_0^{\gamma(\Omega)} z^{-2+1/s} \left| \frac{\partial Z_+}{\partial r} \right|^2 dr dz \\ + \int_0^\infty \int_0^{\gamma(\Omega)} p(r)^{-1} \left| \frac{\partial Z_+}{\partial z} \right|^2 dr dz \leq 0, \end{aligned}$$

namely,

$$\int_0^\infty \int_0^{\gamma(\Omega)} z^{-2+1/s} \left| \frac{\partial Z_+}{\partial r} \right|^2 dr dz + \int_0^\infty \int_0^{\gamma(\Omega)} p(r)^{-1} \left| \frac{\partial Z_+}{\partial z} \right|^2 dr dz \leq 0.$$

Thus $Z_+ \equiv 0$ and the concentration comparison inequality (3.4) follows. \square

4. REGULARITY ESTIMATES

We first introduce the Zygmund spaces, which appear naturally in the regularity scale for solutions to elliptic equations with Gaussian measure in the local setting, see [20]. We refer the reader to the monograph [8] for details about all the related properties we will use for our purposes.

Definition 4.1 (Zygmund spaces). *Let $1 \leq p < \infty$ and $\alpha \in \mathbb{R}$. The Zygmund space $L^p(\log L)^\alpha(\Omega, \gamma)$ is defined as the space of all measurable functions $u : \Omega \rightarrow \mathbb{R}$ such that*

$$\int_\Omega [|u(x)| \log^\alpha(2 + |u(x)|)]^p d\gamma(x) < \infty.$$

If $\alpha = 0$ the Zygmund space $L^p(\log L)^0(\Omega, \gamma)$ coincides with the weighted space $L^p(\Omega, \gamma)$. Moreover, if $p > q$ and $\alpha, \beta \in \mathbb{R}$ then

$$L^p(\log L)^\alpha(\Omega, \gamma) \subset L^q(\log L)^\beta(\Omega, \gamma).$$

When $p = q$ and $\alpha > \beta$ one can prove that

$$(4.1) \quad L^p(\log L)^\alpha(\Omega, \gamma) \subset L^p(\log L)^\beta(\Omega, \gamma).$$

Remark 4.2. The Zygmund space $L^p(\log L)^\alpha(\Omega, \gamma)$ can be equivalently defined as the space of measurable functions $u : \Omega \rightarrow \mathbb{R}$ such that the quantity (which is a quasi-norm in this space)

$$(4.2) \quad \left(\int_0^{\gamma(\Omega)} [(1 - \log t)^\alpha u^\otimes(t)]^p dt \right)^{1/p}$$

is finite. Moreover, $L^p(\log L)^\alpha(\Omega, \gamma)$ is a Banach space when equipped with the norm

$$(4.3) \quad \|u\|_{L^p(\log L)^\alpha(\Omega, \gamma)}^p = \int_0^{\gamma(\Omega)} [(1 - \log t)^\alpha u^{\otimes\otimes}(t)]^p dt,$$

where

$$u^{\otimes\otimes}(t) := \frac{1}{t} \int_0^t u^\otimes(\sigma) d\sigma.$$

We stress that the quasi-norm (4.2) is equivalent to the norm (4.3), see [8] for more details.

The main result of this section is the following regularity result for solutions to the fractional nonlocal OU problem (1.1) in terms of the data f . Notice that when $s = 1$ we recover the corresponding regularity results for the OU equation via Gaussian symmetrization contained in [20].

Theorem 4.3 (Regularity estimates). *Let Ω be an open subset of \mathbb{R}^n , $n \geq 2$, such that $\gamma(\Omega) \leq 1/2$. Fix $0 < s < 1$. If $f \in L^p(\log L)^\alpha(\Omega, \gamma)$, where $\alpha \in \mathbb{R}$ for $2 < p < \infty$, and $\alpha \geq -\frac{s}{2}$ for $p = 2$, then the solution u to (1.1) belongs to $L^p(\log L)^{\alpha+s}(\Omega, \gamma)$ and*

$$\|u\|_{L^p(\log L)^{\alpha+s}(\Omega, \gamma)} \leq C \|f\|_{L^p(\log L)^\alpha(\Omega, \gamma)},$$

for a positive constant $C = C(n, p, \alpha, s, \gamma(\Omega))$ which is independent on u and f .

In order to prove Theorem 4.3 we will first show that the space $\mathcal{H}^s(\Omega, \gamma)$ is embedded in the Zygmund space $L^2(\log L)^{s/2}(\Omega, \gamma)$. This will allow us to choose the datum f in the dual space $L^2(\log L)^{-s/2}(\Omega, \gamma)$ in problem (1.1). In this way Definition 2.5 will still make sense and $u = w(\cdot, 0)$, where w is the solution to (1.6), will be the unique weak solution to problem (1.1). Towards this end we introduce the fractional Gaussian Sobolev space $H^s(\Omega, \gamma)$ as the real interpolation space defined by

$$H^s(\Omega, \gamma) = [H^1(\Omega, \gamma), L^2(\Omega, \gamma)]_{1-s}.$$

Lemma 4.4. *For any $u \in \mathcal{H}^s(\Omega, \gamma)$ the following inequality holds*

$$(4.4) \quad \int_0^{\gamma(\Omega)} [(1 - \log r)^{s/2} u^\otimes(r)]^2 dr \leq C \|u\|_{\mathcal{H}^s(\Omega, \gamma)}^2$$

where C is a positive constant depending on n, s and Ω . In particular,

$$\mathcal{H}^s(\Omega, \gamma) \hookrightarrow L^2(\log L)^{s/2}(\Omega, \gamma).$$

Proof. Given any function $u \in \mathcal{H}^s(\Omega, \gamma)$ we consider the extension \tilde{u} of u by zero outside of Ω . Since $\tilde{u} \in H^s(\mathbb{R}^n, \gamma)$ and this last space coincides with the Gaussian Besov space $B^s(\mathbb{R}^n, \gamma)$ (see [33]), the embedding result contained in [30, Theorem 23] yields

$$\int_0^{1/2} [(1 - \log r)^{s/2} u^\otimes(r)]^2 dr \leq C \|\tilde{u}\|_{H^s(\mathbb{R}^n, \gamma)}^2,$$

for some constant $C > 0$. A change of variable and the monotonicity of the decreasing rearrangement u^\otimes lead to

$$(4.5) \quad \int_0^1 [(1 - \log r)^{s/2} u^\otimes(r)]^2 dr \leq 2 \int_0^{1/2} [(1 - \log r)^{s/2} u^\otimes(r)]^2 dr \leq 2C \|\tilde{u}\|_{H^s(\mathbb{R}^n, \gamma)}^2.$$

Now we observe that the Exact Interpolation Theorem (see [1, Theorem 7.23]) implies that extending any function $u \in \mathcal{H}^s(\Omega, \gamma)$ by zero outside of Ω defines a continuous extension map between $\mathcal{H}^s(\Omega, \gamma)$ and $H^s(\mathbb{R}^n, \gamma)$. Thus it follows that the norm at the right-hand side of (4.5) is bounded (up to a constant depending on n, s and Ω) by $\|u\|_{\mathcal{H}^s(\Omega, \gamma)}^2$ and the result follows. \square

With these results at hand, we are able to show the generalization of the comparison result (Theorem 3.1) for f in Zygmund spaces.

Corollary 4.5. *Assume that $f \in L^2(\log L)^{-s/2}(\Omega, \gamma)$. Then Theorem 3.1 still holds.*

Proof. Let f_n be a sequence of smooth function such that $f_n \rightarrow f$ strongly in $L^2(\log L)^{-s/2}(\Omega, \gamma)$. Let w_n be the unique weak solution to problem (1.6) with data f_n . By choosing w_n as a test function in (2.19) we have

$$\begin{aligned} \iint_{\mathcal{C}_\Omega} y^a |\nabla_{x,y} w_n|^2 d\gamma(x) dy &= c_s^{-1} \int_{\Omega} f_n(x) w_n(x, 0) d\gamma(x) \\ &\leq c_s^{-1} \|w_n(x, 0)\|_{L^2(\log L)^{s/2}(\Omega, \gamma)} \|f_n(x, 0)\|_{L^2(\log L)^{-s/2}(\Omega, \gamma)}. \end{aligned}$$

Next we use (4.4) and the trace inequality (2.18) to find

$$\iint_{\mathcal{C}_\Omega} y^a |\nabla_{x,y} w_n|^2 d\gamma(x) dy \leq C \|w_n\|_{H_{0,L}^1(\mathcal{C}_\Omega, d\gamma(x) \otimes y^a dy)} \|f_n(x, 0)\|_{L^2(\log L)^{-s/2}(\Omega, \gamma)}.$$

This allows us to extract a subsequence from $\{w_n\}$ (still labeled by $\{w_n\}$), such that $w_n \rightarrow w$ weakly in $H_{0,L}^1(\mathcal{C}_\Omega, d\gamma(x) \otimes y^a dy)$. Then the compact embedding established in Proposition 2.4 gives that, up to a new subsequence, $w_n(\cdot, 0) \rightarrow w(\cdot, 0)$ strongly in $L^2(\Omega, \gamma)$. Thus we can pass to the limit in the weak formulation (2.19) of w_n and find that w solves problem (1.6) corresponding to the data f . Thus $u := w(\cdot, 0)$ is the weak solution to problem (1.1). In order to obtain the concentration inequality (3.1), we just observe that f_n^\star approximates f^\star in $L^2(\Omega^\star, \gamma)$. Then, if $\{w_n\}$ and $\{v_n\}$ are sequences of approximating solutions converging to w and v respectively, passing to the limit in the integral inequality

$$\int_0^s w_n^\otimes(\sigma, 0) d\sigma \leq \int_0^s v_n^\otimes(\sigma, 0) d\sigma,$$

we immediately get (3.1). \square

For the proof of Theorem 4.3 we need two further preliminary results, interesting in their own right. The following is a regularity result for solutions of problems of the type (1.1) with *rearranged* data, posed on the half-space H .

Theorem 4.6 (Estimates for half-space solutions). *Let $H = \{x \in \mathbb{R}^n : x_1 > 0\}$. Suppose that $h(x) = h^\star(x)$, for all $x \in H$. If $h \in L^p(\log L)^\alpha(H, \gamma)$ with $\alpha \in \mathbb{R}$ for $2 < p < \infty$, and $\alpha \geq -\frac{s}{2}$ for $p = 2$, then the weak solution ψ to*

$$(4.6) \quad \begin{cases} \mathcal{L}^s \psi = h, & \text{in } H, \\ \psi = 0, & \text{on } \partial H, \end{cases}$$

belongs to $L^p(\log L)^{\alpha+s}(H, \gamma)$ and

$$\|\psi\|_{L^p(\log L)^{\alpha+s}(H, \gamma)} \leq C \|h\|_{L^p(\log L)^\alpha(H, \gamma)},$$

for some constant $C = C(n, p, \alpha, s) > 0$, which is independent on ψ and h .

Proof. By (2.21) and (2.9)–(2.11) we can write

$$\psi(x) = \frac{1}{\Gamma(s)} \int_0^\infty e^{-t(\mathcal{L}_H)} h(x) \frac{dt}{t^{1-s}} = \frac{1}{\Gamma(s)} \int_0^\infty e^{-t\mathcal{L}} \tilde{h}(x) \frac{dt}{t^{1-s}} = \mathcal{L}^{-s} \tilde{h}(x).$$

Then the estimate follows from [29, Theorem 5.7]. \square

The next Lemma is a useful comparison principle for solutions of problems of the form (1.1) with rearranged data, having as a ground domain an half-space of Gaussian measure larger than $1/2$.

Lemma 4.7 (Comparison of half-space solutions). *Let $H_\omega = \{x \in \mathbb{R}^n : x_1 > \omega\}$, for some $\omega > 0$. Let $h \in L^p(\log L)^\alpha(H, \gamma)$ be a nonnegative function such that $h(x) = h^\star(x)$ and let ψ be the weak solution to*

$$\begin{cases} \mathcal{L}^s \psi = h, & \text{in } H_\omega, \\ \psi = 0, & \text{on } \partial H_\omega. \end{cases}$$

Then

$$\psi(x) \leq \zeta(x), \quad \text{for a.e. } x \in H_\omega,$$

where ζ is the weak solution to (4.6) with datum \bar{h} , where \bar{h} denotes the zero extension of h in $H \setminus H_\omega$.

Proof. The function

$$F(x, t) := e^{-t(\mathcal{L}_H)} \bar{h}(x) - e^{-t(\mathcal{L}_{H_\omega})} h(x),$$

solves the initial boundary value problem

$$\begin{cases} \partial_t F = \Delta F - x \cdot \nabla F, & \text{in } H_\omega \times (0, \infty), \\ F(x, t) \geq 0, & \text{on } \partial H_\omega \times (0, \infty), \\ F(x, 0) = 0, & \text{on } H_\omega. \end{cases}$$

Thus, by a standard maximum principle argument, $F \geq 0$ in $H_\omega \times [0, \infty)$. In other words,

$$e^{-t(\mathcal{L}_H)} \bar{h} \geq e^{-t(\mathcal{L}_{H_\omega})} h, \quad \text{for all } x \in H_\omega, \quad t \geq 0.$$

Therefore, if v and \bar{v} denote the extensions as in (2.16) of ψ and ζ , respectively, then

$$\bar{v}(x, y) \geq v(x, y), \quad \text{for all } x \in H_\omega, \quad y \geq 0.$$

The result follows by taking $y = 0$ in this last inequality. \square

Now we are finally able to present the proof of the regularity estimate, namely, Theorem 4.3.

Proof of Theorem 4.3. Let u be the weak solution to (1.1) defined in an open set Ω such that $\gamma(\Omega) \leq 1/2$, with corresponding datum f . By Theorem 3.1, u is less concentrated than the solution ψ to (1.5) defined in the half-space with the same Gauss measure as Ω and datum f^\star . If $\gamma(\Omega) = 1/2$ the assertion follows by Theorem 4.6. If $\gamma(\Omega) < 1/2$, we first apply Lemma 4.7 to estimate ψ in terms of the solution ζ to (4.6) defined in the half-space $H = \{x \in \mathbb{R}^n : x_1 > 0\}$ and having the extension of f^\star by zero to H at the right-hand side. Then Theorem 4.6 allows us to conclude. \square

5. APPENDIX: A SEMIGROUP LANGUAGE PROOF OF THE L^p ESTIMATE

For completeness and convenience of the reader, we give an alternative and more explicit proof of Theorem 4.6 with L^p data using the Mehler kernel to represent the inverse of the fractional OU operator. Observe that such result is a particular case of Theorem 4.6 since, when $f \in L^p(\Omega, \gamma)$, Theorem 4.6 and the embedding (4.1) give $u \in L^p(\log L)^s(\Omega, \gamma) \subset L^p(\Omega, \gamma)$.

Theorem 5.1 (Estimates for half-space solutions with L^p data). *Let $H = \{x \in \mathbb{R}^n : x_1 > 0\}$. Suppose that $h(x) = h^\star(x)$, for all $x \in H$. If $h \in L^p(H, \gamma)$, for $2 \leq p < \infty$, then the weak solution ψ to (4.6) belongs to $L^p(H, \gamma)$ and*

$$\|\psi\|_{L^p(H, \gamma)} \leq C \|h\|_{L^p(H, \gamma)},$$

for some constant $C = C(n, p, s) > 0$, which is independent on ψ and h .

Proof. The proof will be split in four steps.

Step 1. The explicit solution via the semigroup kernel. By (2.21), and by using an abuse of notation, the solution ψ to (4.6) can be written as

$$\psi(x) = \psi(x_1) = \frac{1}{\Gamma(s)} \int_0^\infty e^{-t(\mathcal{L}_H)} h(x) \frac{dt}{t^{1-s}} = \int_0^\infty G(x_1, y_1) h(y_1) d\gamma(y_1),$$

where (see (2.12))

$$G(x_1, y_1) = \frac{1}{\Gamma(s)} \int_0^\infty [M_t(x_1, y_1) - M_t(x_1, -y_1)] \frac{dt}{t^{1-s}}.$$

Next we write

$$\begin{aligned} (5.1) \quad G(x_1, y_1) &= \int_0^{c(p)} \cdots dt + \int_{c(p)}^{T(x_1, y_1)} \cdots dt + \int_{T(x_1, y_1)}^\infty \cdots dt \\ &=: G_1(x_1, y_1) + G_2(x_1, y_1) + G_3(x_1, y_1), \end{aligned}$$

with $c(p) > 1$ a suitable constant, and $T(x_1, y_1) = \max\{c(p), \log(x_1^2 + y_1^2)\}$. It follows that

$$(5.2) \quad \|\psi\|_{L^p(H, \gamma)}^p \leq \sum_{j=1}^3 \int_0^\infty \left(\int_0^\infty G_j(x_1, y_1) h(y_1) d\gamma(y_1) \right)^p d\gamma(x_1).$$

Step 2. Estimate of the term $j = 1$ in (5.2). We observe that by (2.7) and (2.10) we get

$$(5.3) \quad \left\| \int_{-\infty}^\infty M_t(x_1, y_1) \tilde{h}(y_1) d\gamma(y_1) \right\|_{L^p(\mathbb{R}, \gamma)} \leq \|\tilde{h}\|_{L^p(\mathbb{R}, \gamma)} = 2 \|h\|_{L^p(H, \gamma)},$$

where \tilde{h} is defined like in (2.9). Tonelli's theorem, Minkowski's inequality and (5.3) yield

$$\begin{aligned} & \left\| \int_0^\infty G_1(x_1, y_1) h(y_1) d\gamma(y_1) \right\|_{L^p(H, \gamma)} \\ & \leq c_s \int_0^{c(p)} \left\| \int_0^\infty [M_t(x_1, y_1) - M_t(x_1, -y_1)] h(y_1) d\gamma(y_1) \right\|_{L^p(H, \gamma)} \frac{dt}{t^{1-s}} \\ & = c_s \int_0^{c(p)} \left\| \int_{-\infty}^\infty M_t(x_1, y_1) \tilde{h}(y_1) d\gamma(y_1) \right\|_{L^p(\mathbb{R}, \gamma)} \frac{dt}{t^{1-s}} \\ & \leq 2c_s \|h\|_{L^p(H, \gamma)} \int_0^{c(p)} \frac{dt}{t^{1-s}} = c_{s,p} \|h\|_{L^p(H, \gamma)}. \end{aligned}$$

Step 3. Estimate of G_2 and G_3 . We prove that

$$\int_0^\infty \left(\int_0^\infty G_j^{p'}(x_1, y_1) d\gamma(y_1) \right)^{p/p'} d\gamma(x_1) < \infty, \quad \text{for } j = 2, 3.$$

By Jensen's inequality, it is enough to show that $G_j \in L^p(H \times H, \gamma \otimes \gamma)$, for $j = 2, 3$. If $t > c(p) > 1$ then $(1 - e^{-2t}) \sim 1$ and $|M_t(x_1, y_1)| \leq c \exp(4e^{-t}|x_1||y_1|)$, see (2.5). It follows that

$$\begin{aligned} |G_2(x_1, y_1)| & \leq c_s \int_{c(p)}^{T(x_1, y_1)} |M_t(x_1, y_1) - M_t(x_1, -y_1)| \frac{dt}{t^{1-s}} \\ & \leq \frac{c_s}{c(p)^{1-s}} \int_{c(p)}^{T(x_1, y_1)} \exp(4e^{-t}|x_1||y_1|) dt \\ & \leq \frac{c_s}{c(p)^{1-s}} \int_{c(p)}^{T(x_1, y_1)} \exp\left[2e^{-c(p)}(x_1^2 + y_1^2)\right] dt \\ & \leq \frac{c_s}{c(p)^{1-s}} \cdot \frac{T(x_1, y_1)}{(\varphi(x_1))^{4e^{-c(p)}} (\varphi(y_1))^{4e^{-c(p)}}} =: \tilde{G}_2(x_1, y_1). \end{aligned}$$

We then get $\tilde{G}_2(x_1, y_1) \in L^p(H \times H, \gamma \otimes \gamma)$ if we choose $4pe^{-c(p)} < 1$, that is, if $c(p) > \max\{1, \log(4p)\}$. Moreover, by Taylor's formula and using that $t > 1$ and $e^{-t}(|x_1|^2 + |y_1|^2) < 1$,

$$\begin{aligned} M_t(x_1, y_1) - M_t(x_1, -y_1) & \leq C_n \left| \exp\left(\frac{e^{-t}\langle x_1, y_1 \rangle}{1 - e^{-2t}}\right) - \exp\left(-\frac{e^{-t}\langle x_1, y_1 \rangle}{1 - e^{-2t}}\right) \right| \\ & \leq C e^{-t} |\langle x_1, y_1 \rangle| \exp(ce^{-t}\langle x_1, y_1 \rangle) \\ & \leq C e^{-t} (|x_1|^2 + |y_1|^2) \exp(ce^{-t}(|x_1|^2 + |y_1|^2)) \\ & \leq C e^{-t} (|x_1|^2 + |y_1|^2). \end{aligned}$$

Then

$$\begin{aligned} |G_3(x_1, y_1)| & \leq C_s \int_{T(x_1, y_1)}^\infty |M_t(x_1, y_1) - M_t(x_1, -y_1)| \frac{dt}{t^{1-s}} \leq C_{n,s} (|x_1|^2 + |y_1|^2) \int_{T(x_1, y_1)}^{+\infty} e^{-t} dt \\ & = C_{n,s} (|x_1|^2 + |y_1|^2) e^{-T(x_1, y_1)} \leq C_{n,s} \in L^p(H \times H, \gamma \otimes \gamma). \end{aligned}$$

Step 4. Estimates of the terms $j = 2, 3$ in (5.2). By Hölder's inequality and the estimates of Step 3, we get

$$\begin{aligned} & \int_0^\infty \left(\int_0^\infty G_j(x_1, y_1) h(y_1) d\gamma(y_1) \right)^p d\gamma(x_1) \\ & \leq \int_0^\infty \left(\int_0^\infty G_j^{p'}(x_1, y_1) d\gamma(y_1) \right)^{p/p'} \left(\int_0^\infty |h(y_1)|^p d\gamma(y_1) \right) d\gamma(x_1) \leq c \|h\|_{L^p(H, \gamma)}^p, \end{aligned}$$

for $j = 2, 3$ and for some positive constant $c = c(n, p, s)$.

Hence the desired result follows by collecting Steps 2 and 4 in estimate (5.2). \square

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REFERENCES

- [1] R. A. ADAMS AND J. J. F. FOURNIER, *Sobolev Spaces*, vol. 140 of Pure and Applied Mathematics (Amsterdam), Elsevier/Academic Press, Amsterdam, second ed., 2003.
- [2] M. ALLEN, *A fractional free boundary problem related to a plasma problem*, arXiv:1507.06289 (2015), 13pp.
- [3] A. ALVINO, G. TROMBETTI, J. I. DIAZ, AND P. L. LIONS, *Elliptic equations and Steiner symmetrization*, Comm. Pure Appl. Math., 49 (1996), pp. 217–236.
- [4] A. ALVINO, G. TROMBETTI, AND P.-L. LIONS, *Comparison results for elliptic and parabolic equations via Schwarz symmetrization*, Ann. Inst. H. Poincaré Anal. Non Linéaire, 7 (1990), pp. 37–65.
- [5] A. ANDERSSON AND P. SJÖGREN, *Ornstein-Uhlenbeck theory in finite dimension*, Preprint 2012:12, Matematiska vetenskaperna, Göteborg 2012, 40pp.
- [6] D. APPLEBAUM, *Lévy Processes and Stochastic Calculus*, Cambridge Studies in Advanced Mathematics, vol. 116, Second Edition, Cambridge University Press, Cambridge, UK, 2009.
- [7] C. BANDLE, *On symmetrizations in parabolic equations*, J. Analyse Math., 30 (1976), pp. 98–112.
- [8] C. BENNETT AND R. SHARPLEY, *Interpolation of Operators*, vol. 129 of Pure and Applied Mathematics, Academic Press, Inc., Boston, MA, 1988.
- [9] M. F. BETTA, F. BROCK, A. MERCALDO, AND M. R. POSTERARO, *A comparison result related to Gauss measure*, C. R. Math. Acad. Sci. Paris, 334 (2002), pp. 451–456.
- [10] M. F. BETTA, F. CHIACCHIO AND A. FERONE, *Isoperimetric estimates for the first eigenfunction of a class of linear elliptic problems*, Z. angew. Math. Phys. 58 (2007), pp. 37–52.
- [11] V. I. BOGACHEV, *Gaussian Measures*, vol. 62 of Mathematical Surveys and Monographs, American Mathematical Society, Providence, RI, 1998.
- [12] M. BONFORTE, Y. SIRE, AND J. L. VÁZQUEZ, *Existence, uniqueness and asymptotic behaviour for fractional porous medium equations on bounded domains*, Discrete Contin. Dyn. Syst., 35 (2015), pp. 5725–5767.
- [13] C. BORELL, *The Brunn-Minkowski inequality in Gauss space*, Invent. Math., 30 (1975), pp. 207–216.
- [14] L. A. CAFFARELLI AND L. SILVESTRE, *An extension problem related to the fractional Laplacian*, Comm. Partial Differential Equations, 32 (2007), pp. 1245–1260.
- [15] L. A. CAFFARELLI AND Y. SIRE, *On some pointwise inequalities involving nonlocal operators*, arXiv:1604.05665 (2016), 17pp.
- [16] L. A. CAFFARELLI AND P. R. STINGA, *Fractional elliptic equations, Caccioppoli estimates and regularity*, Ann. Inst. H. Poincaré Anal. Non Linéaire, 33 (2016), pp. 767–807.
- [17] F. CHIACCHIO, *Comparison results for linear parabolic equations in unbounded domains via Gaussian symmetrization*, Differential Integral Equations, 17 (2004), pp. 241–258.
- [18] K. M. CHONG, *Some extensions of a theorem of Hardy, Littlewood and Pólya and their applications*, Canad. J. Math., 26 (1974), pp. 1321–1340.
- [19] K. M. CHONG AND N. M. RICE, *Equimeasurable rearrangements of functions*, Queen’s University, Kingston, Ont., 1971. Queen’s Papers in Pure and Applied Mathematics, No. 28.
- [20] G. DI BLASIO, F. FEO, AND M. R. POSTERARO, *Regularity results for degenerate elliptic equations related to Gauss measure*, Math. Inequal. Appl., 10 (2007), pp. 771–797.
- [21] G. DI BLASIO AND B. VOLZONE, *Comparison and regularity results for the fractional Laplacian via symmetrization methods*, J. Differential Equations, 253 (2012), pp. 2593–2615.
- [22] A. EHRHARD, *Inégalités isopérimétriques et intégrales de Dirichlet gaussiennes*, Ann. Sci. École Norm. Sup. (4), 17 (1984), pp. 317–332.
- [23] H. FEDERER, *Geometric Measure Theory*, Die Grundlehren der mathematischen Wissenschaften, Band 153, Springer-Verlag New York Inc., New York, 1969.
- [24] F. FEO AND M. R. POSTERARO, *Logarithmic Sobolev trace inequalities*, Asian J. Math., 17 (2013), pp. 569–582.

- [25] V. FERONE AND A. MERCALDO, *A second order derivation formula for functions defined by integrals*, C. R. Acad. Sci. Paris Sér. I Math., 326 (1998), pp. 549–554.
- [26] J. E. GALÉ, P. J. MIANA AND P. R. STINGA, *Extension problem for fractional operators: semigroups and wave equations*, J. Evol. Equ., 13 (2013), pp. 343–368.
- [27] P. HAJLASZ, *Sobolev mappings, co-area formula and related topics*, in Proceedings on Analysis and Geometry (Russian) (Novosibirsk Akademgorodok, 1999), Izdat. Ross. Akad. Nauk Sib. Otd. Inst. Mat., Novosibirsk, 2000, pp. 227–254.
- [28] Y. HASHIMOTO, *A remark on the analyticity of the solutions for non-linear elliptic partial differential equations*, Tokyo J. Math., 29 (2006), pp. 271–281.
- [29] G. E. KARADZHOV AND M. MILMAN, *Extrapolation theory: new results and applications*, J. Approx. Theory, 133 (2005), pp. 38–99.
- [30] J. MARTÍN AND M. MILMAN, *Fractional Sobolev inequalities: symmetrization, isoperimetry and interpolation*, Astérisque, (2014), pp. x+127.
- [31] V. G. MAZ'JA, *Weak solutions of the Dirichlet and Neumann problems*, Trudy Moskov. Mat. Obšč., 20 (1969), pp. 137–172.
- [32] J. MOSSINO AND J.-M. RAKOTOSON, *Isoperimetric inequalities in parabolic equations*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4), 13 (1986), pp. 51–73.
- [33] E. V. NIKITIN, *Comparison of two definitions of Besov classes on infinite-dimensional spaces*, Math. Notes, 95 (2014), pp. 133–135. Translation of Mat. Zametki 95 (2014), no. 1, 150–153.
- [34] M. NOVAGA, D. PALLARA, AND Y. SIRE, *A fractional isoperimetric problem in the Wiener space*, to appear in J. Anal. Math.
- [35] M. NOVAGA, D. PALLARA, AND Y. SIRE, *A symmetry result for degenerate elliptic equations on the Wiener space with nonlinear boundary conditions and applications*, Discrete Contin. Dyn. Syst. Ser. S, 9 (2016), pp. 815–831.
- [36] E. PRIOLA, *On a Dirichlet problem involving an Ornstein-Uhlenbeck operator*, Potential Anal., 18 (2003), pp. 251–287.
- [37] J. M. RAKOTOSON AND B. SIMON, *Relative rearrangement on a finite measure space. Application to the regularity of weighted monotone rearrangement. I*, Rev. R. Acad. Cienc. Exactas Fís. Nat. (Esp.), 91 (1997), pp. 17–31.
- [38] Y. SIRE, J. L. VÁZQUEZ, AND B. VOLZONE, *Symmetrization for fractional elliptic and parabolic equations and an isoperimetric application*, to appear in Chin. Ann. Math.
- [39] R. SONG AND Z. VONDRAČEK, *Potential theory of subordinate killed Brownian motion in a domain*, Probab. Theory Related Fields, 125 (2003), pp. 578–592.
- [40] P. R. STINGA AND J. L. TORREA, *Extension problem and Harnack's inequality for some fractional operators*, Comm. Partial Differential Equations, 35 (2010), pp. 2092–2122.
- [41] P. R. STINGA AND B. VOLZONE, *Fractional semilinear Neumann problems arising from a fractional Keller–Segel model*, Calc. Var. Partial Differential Equations, in press (2015).
- [42] P. R. STINGA AND C. ZHANG, *Harnack's inequalities for fractional nonlocal equations*, Discrete Contin. Dyn. Syst. 33 (2013), 3153–3170.
- [43] G. TALENTI, *Elliptic equations and rearrangements*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4), 3 (1976), pp. 697–718.
- [44] J. L. VÁZQUEZ, *Symmetrization and mass comparison for degenerate nonlinear parabolic and related elliptic equations*, Adv. Nonlinear Stud., 5 (2005), pp. 87–131.
- [45] J. L. VÁZQUEZ AND B. VOLZONE, *Symmetrization for linear and nonlinear fractional parabolic equations of porous medium type*, J. Math. Pures Appl. (9), 101 (2014), pp. 553–582.
- [46] J. L. VÁZQUEZ AND B. VOLZONE, *Optimal estimates for fractional fast diffusion equations*, J. Math. Pures Appl. (9), 103 (2015), pp. 535–556.
- [47] B. VOLZONE, *Symmetrization for fractional Neumann problems*, Nonlinear Anal., 147 (2016), pp. 1–25.
- [48] H. F. WEINBERGER, *Symmetrization in uniformly elliptic problems*, in Studies in mathematical analysis and related topics, Stanford Univ. Press, Stanford, Calif., 1962, pp. 424–428.

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